Sampling theorem I

- In the fourier transform, sometimes the notation ω is substituted by $2\pi f$:

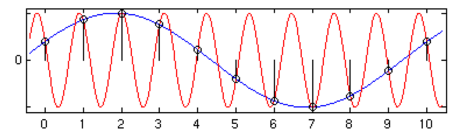
$$X(f) = \int_{-\infty}^{\infty} x(t)\ e^{-i2\pi f t}\ dt$$

and the inverse Fourier becomes:

$$x(t) = \int_{-\infty}^{\infty} X(f)\ e^{i2\pi f t}\ df$$

- In most situation, you observe a function $x(t)$ at regular recording interval Δ i.e. you have a set of discrete observations $x(n\Delta)$ with $n \in \mathbb{Z}$. how can we recover $x(t)$ when having a set of discrete observations recorded at regular intervals?

Sampling theorem II



Example of aliasing: From the same set of samples, two different sinusoids can be recovered. The one in red as a higher frequency than the one in blue. So there is a possible ambiguity when we want to recover the original signal. The sampling theorem proposes to recover exactly $x(t)$ in some particular case.

Sampling theorem III

Definition (Fourier series)

Let $u(t)$ be a function time limited on $[\tau, \tau+T]$, then u can be expanded:

$$u(t) = \sum_{n=-\infty}^{\infty} u_n\ \exp\left(\frac{\imath 2\pi n t}{T}\right), \quad \tau \leq t \leq \tau + T$$

where the Fourier series coefficients u_n are given by:

$$u_n = \frac{1}{T}\int_{\tau}^{\tau+T} u(t) \exp\left(\frac{-\imath 2\pi n t}{T}\right) dt$$

Sampling theorem IV

Definition (Nyquist critical frequency)

For any sampling interval Δ, the Nyquist critical frequency is defined as:

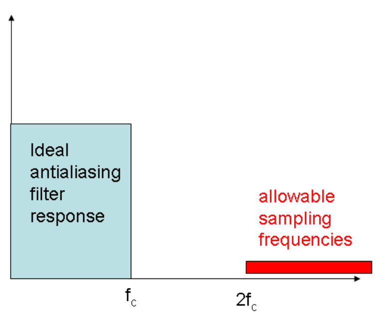
$$f_c = \frac{1}{2\Delta}$$

Theorem (Sampling theorem)

If a continuous function $x(t)$, sampled at interval Δ, happens to be bandwidth limited to frequencies smaller than f_c (i.e $X(f) = 0 \quad \forall f > f_c$), then x can be exactly recovered from its samples $\{x(n\Delta)\}_{n\in\mathbb{Z}}$:

$$x(t) = \sum_{n=-\infty}^{+\infty} x(n\Delta)\ sinc\left(\frac{t - n\Delta}{\Delta}\right)$$

Sampling theorem V



Choosing the sampling frequency.

Sampling theorem VI

Proof:

- Lets assume we have *a continuous function $x(t)$, sampled at interval Δ, happens to be bandwidth limited to frequencies smaller than f_c (i.e $X(f) = 0 \quad \forall f > f_c$).*
- Compute the Fourier series expansion of $X(f)$:

$$X(f) = \sum_{n=-\infty}^{+\infty} X_n \exp(-\frac{\imath 2\pi n f}{2 f_c})$$

- Taking the inverse Fourier transform (between $-f_c$ and f_c since $X(f)$ is zero elsewhere) gives:

$$x(t) = \sum_{n=-\infty}^{+\infty} 2 f_c X_n \mathrm{sinc}(2 f_c t - n)$$

Sampling theorem VII

- at $t = n/2f_c$ we have $x(n/2f_c) = 2f_c X_n$ so we can replace:

$$x(t) = \sum_{n=-\infty}^{+\infty} x(n/2f_c)\text{sinc}(2f_c t - n)$$

with $x(n/2f_c)$ the values of $x(t)$ that have been sampled at $\Delta = \frac{1}{2f_c}$.

Examples of the use of the sampling theorem

- The audible frequency range of 20Hz - 20kHz. To encode the highest frequency, we need a sampling rate $f_s \leq 2 \times 20kHz$. The CD is sampled at 44.1 kHz.
- The voice frequency band is below 4000 Hz. The sampling frequency used for the phone is then 8000 Hz.
- What about the sampling frequency of motion capture data ?