Introduction

- PDE: definition, properties, applications.
- Solving PDE analytically.
- Solving PDE numerically: Finite Difference Method.

Partial Differential Equations (PDE)

Definition

Partial differential equations (PDEs) are differential equations where the unknown function (solution) is a function of several independent variables.
In general, a PDE in one spatial variable x and one temporal variable t is an equation of the form:

$$G(x,t,u,u_x,u_t,u_{xx},u_{tt},u_{xt},\cdots)=0 \quad x\in\Omega, \quad t\in I$$

where I is the interval of time, Ω in the spatial domain, and $u=u(x,t)$ is the unknown function that possesses as many continuous partial derivatives as required by G.

Applications of PDEs

- Quantum Mechanics
- Thermodynamics (Heat flows)
- Electromagnetism (Maxwell equations)
- Computer graphics
- Computer vision
- etc.

Linearity of PDEs I

Definition

The PDE:

$$G(x,t,u,u_x,u_t,u_{xx},u_{tt},u_{xt},\cdots)=0 \quad x\in\Omega, \quad t\in I$$

is linear if G is a linear function in u and in all of its derivatives. It means that the unknown u and all its derivatives that are present appear alone and to the first power in the equation.
Otherwise, it is nonlinear.

Linearity of PDEs II

Example (Are those PDEs linear or nonlinear?)

- $u_t + u\ u_{xx} = 0$
- $u_{tt} - u_x + \sin u = 0$
- $u_t - \sin(x^2 t)\ u_{xx} = 0$

Linearity of PDEs III

- Linear equations have an algebraic structure to their solution sets; for example, the sum of two solutions to a homogeneus linear equation is again a solution, as are constant multiples of solutions.
- Nonlinear equations do not share this property.

Boundary and initial conditions

Example of the Diffusion equation.

$$u_t = k u_{xx}$$

The boundary conditions imposes the values of the function u on the boundary of the spatial domain:

$$u(0,t) = 0, \quad u(l,t) = 0, \quad \forall t \geq 0$$

The initial condition imposes the values of u at $t = 0$:

$$u(x,0) = \phi(x), \quad 0 < x < l$$

Laplacian

Definition

Lets consider the function $u(x_1, x_2, \cdots, x_n)$, its laplacian is defined as:

$$\Delta u = \sum_i u_{x_i x_i}$$

Exercises on PDEs

Example (Solving $u_x = t \sin x$.)

By direct integration $u(x,t) = -t\cos x + \phi(t)$ where ϕ is an arbitrary function.

1. Verify that:
$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp^{\frac{-x^2}{4kt}}$$
satisfies the diffusion equation $u_t = k u_{xx}$.

2. Verify that $u(x,y) = \ln\sqrt{x^2+y^2}$ satisfies the PDE:
$$\Delta u = 0$$

Exercise: Diffusion equation I

We consider the problem of signal propagation along a thin wire.

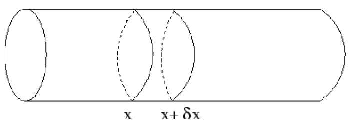
$$rc\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$$

for $V(x,t)$ as a function of both space $(x \geq 0)$ and time $(t \geq 0)$, subject to the following initial and boundary conditions:

$$\begin{aligned} V(x,0) &= 0V, \quad x > 0 \\ V(0,t) &= 5V, \quad t \geq 0 \end{aligned}$$

The length of the thin wire is $l = 2000\mu m$. The resistance is $r = 20\Omega/\mu m$ and $c = 10^{-9} F/m$.

Exercise: Diffusion equation II



Exercise: Diffusion equation III

1. Propose a numerical algorithm to compute $v(x,t)$
2. We want to compute the voltage at 21 locations (including the initial condition x_0). Compute h_x and the locations $\{x_n\}$.
3. Lets $k = \frac{h_t}{rch_x^2}$. Generate the values of v for different values of k. Is there an optimal value of k? Are there bad values of k?
4. Set $k = 0.5$. Verify that $t_n \propto x_n^2$.