Linear Algebra

- yesterday, we looked at appendices A & B
- today, we look at appendices C & D

Exercises

1. Do B.4
2. Do B.8
3. If A and B are invertible matrices, simplify

$$(A^{-1}B^2)^{-1}(BA^{-1})(ABA^{-1})^{-1}(BA)(A^{-1}B).$$

4. Do C.17

Solving Linear Systems I

Knowing the matrix A and the vector $\mathbf{y}$, find the vector $\mathbf{x}$ such that

$$\mathrm{A}\mathbf{x} = \mathbf{y} \tag{1}$$

1. If A is square
   1. and $\det(\mathrm{A}) \neq 0$, then A^{-1} exists. The solution $\mathbf{x}$ is:

      $$\mathbf{x} = \mathrm{A}^{-1}\mathbf{y}$$

   2. If $\det(\mathrm{A}) = 0$, A is singular: at least one of its row and column are dependent of the others. In this case, the linear system has either no solution or an infinite number of solution (c.f. C2.3. page C-7).

Solving Linear Systems II

2. For any matrix A (square or not), you can multiply the original equation on the left side by A^T:

   $$\mathrm{A}^T\mathrm{A}\mathbf{x} = \mathrm{A}^T\mathbf{y} \quad \textit{normal equation} \tag{2}$$

   Then $\mathrm{C} = \mathrm{A}^T\mathrm{A}$ is square, and we are back to case 1 having to solve $\mathrm{C}\mathbf{x} = \mathbf{y}'$ with $\mathbf{y}' = \mathrm{A}^T\mathbf{y}$.
   1. If $\mathrm{A}^T\mathrm{A}$ is non singular, a solution of equation (2) can be proposed:

      $$\mathbf{x}_{LS} = (\mathrm{A}^T\mathrm{A})^{-1}\mathrm{A}^T\mathbf{y}$$

      This solution is called the Least Squares solution and solves equation (2). $(\mathrm{A}^T\mathrm{A})^{-1}\mathrm{A}^T$ is called the *pseudo-inverse* of A.

Solving Linear Systems III

Warning: a solution of equation (1) is a solution of equation (2) but not automatically vice et versa i.e.

$$(1) \Rightarrow (2)$$

but

$$(1) \not\Leftarrow (2)$$

Solving Linear Systems IV

2. If $\mathrm{A}^T\mathrm{A}$ is singular, c.f. C2.3. page C-7

Remarks on $\mathbf{x}_{LS}$

- Q: So how to check that $\mathbf{x}_{LS}$ is a solution of equation (1) ?
  A: Just compute $\mathrm{A}\mathbf{x}_{LS}$ and check that it is equal to $\mathbf{y}$. It is equivalent to show that
  $$\|\mathrm{A}\mathbf{x}_{LS} - \mathbf{y}\| = 0$$
  if $\|\mathrm{A}\mathbf{x}_{LS} - \mathbf{y}\| \neq 0$ then there is no solution $\mathbf{x}$ that satisfies equation (1).
- Q: What is $\mathbf{x}_{LS}$ representing if $\|\mathrm{A}\mathbf{x}_{LS} - \mathbf{y}\| \neq 0$? Any use of $\mathbf{x}_{LS}$ then?
  A: yes, the solution $\mathbf{x}_{LS}$ is interesting because it minimises the norm :
  $$\|\mathrm{A}\mathbf{x}_{LS} - \mathbf{y}\| \leq \|\mathrm{A}\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}$$
  or
  $$\frac{\partial \|\mathrm{A}\mathbf{x} - \mathbf{y}\|}{\partial \mathbf{x}} = 0 \quad \text{if} \quad \mathbf{x} = \mathbf{x}_{LS} \tag{3}$$

Exercises

1. Prove equation (3) using the table in page D-4.
2. We have a set of points $\{(x_i, y_i)\}_{i=1\cdots N}$. These points lie on a quadratic curve with equation
   $$f(x) = y = \theta_2 x^2 + \theta_1 x + \theta_0$$
   How would you compute $\Theta = (\theta_0\ \theta_1\ \theta_2)^T$?