Second Order Differential Equations I

Some second order differrential equations can be solved using substitution:

- For the equation $F(t, y'', y') = 0$, substitute $v = y'$ to reduce to a first order equation
- For the equation $F(y, y', y'') = 0$ (no explicit appearance of the variable t), substitute $v = y'$ to transform to a first order equation in $v = v(y)$: the equation becomes $F(y, v, v\frac{dv}{dy}) = 0$ (1st order equation in $v(y)$).

Linear, Constant-coefficient Equations I

Theorem (Linear, Constant-coefficient Equations)

The equation

$$ay'' + by' + cy = 0$$

where a, b, c are constants, is linear with constant-coefficient, and has solution of the form $y = \exp(mt)$ where m is a root of $am^2 + bm + c = 0$.

Linear, Constant-coefficient Equations II

- case I: m_1 and m_2 are real solutions and unequal. In this case we have two independent solutions: $\exp(m_1 t)$ and $\exp(m_2 t)$.
- case II: $m_1 = m_2$ are reals, and in this case we have two independent solutions: $\exp(m_1 t)$ and $t\exp(m_1 t)$.
- case III: $m_1 = \alpha + \imath\beta$ and $m_2 = \alpha - \imath\beta$, and the two solutions are $\exp(\alpha t)\sin(\beta t)$ and $\exp(\alpha t)\cos(\beta t)$.

Cauchy-Euler Equations I

Theorem (Cauchy-Euler equations)

A Cauchy-Euler equation has the form:

$$at^2y'' + bty' + cy = 0$$

and has solutions of the form $y = t^m$ where m can be determined with the equation:

$$a(m-1)m + bm + c = 0$$

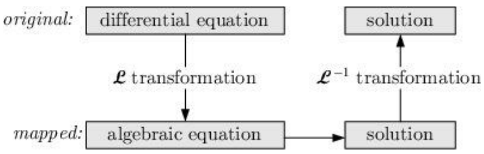
Cauchy-Euler Equations II

- Case I: $m_1 \neq m_2$ are real, then the two independent solutions are $y(t) = t^{m_1}$ and $y(t) = t^{m_2}$.
- Case II: $m_1 = m_2$ are real, then the two independent solutions are $y(t) = t^{m_1}$ and $y(t) = t^{m_2}\ln t$.
- Case III: $m_1 = \alpha + \imath\beta$ and $m_2 = \alpha - \imath\beta$ are complex, then the two independent solutions are: $t^\alpha\sin(\beta\ln t)$ and $t^\alpha\cos(\beta\ln t)$

Exercise 2nd order equations

1. Explain the previous theorem.
2. Solve:
$$y'' + y' + y = 0$$
3. Solve:
$$t^2y'' - 3ty + 4y = 0$$

Solving linear differential equations using the Laplace transform I



http://www.atp.ruhr-uni-bochum.de/rt1/syscontrol/node11.html

Laplace transform

Definition

The one sided Laplace transform of a function $f(t)$ is defined by:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

This transform explicitly includes initial condition at $t=0$ and for this reason is used to model causal systems.

Exercises Laplace transform

1. Compute the Laplace transform of $f(t)=1, \quad t \geq 0.$
2. Compute the Laplace transform of $f(t)=\exp(-t)$.
3. Compute the Laplace transform of $f(t)=t\ \exp(-t)$.

Scalar product for continuous functions

Definition (scalar product)

Consider two functions integrable on an interval $[a,b]$, the scalar (or dot) product is:

$$<f,g>=\int_a^b f(t)\cdot g(t)dt$$

you can verify that it is a scalar product

- $<f,g>=<g,f>$
- $<f+h,g>=<f,g>+<h,g>$
- $<\alpha f,g>=\alpha<f,g>$
- $<f,f>=\|f\|^2 \geq 0$
- if $\|f\|=0$ then $f(t)=0, \quad \forall t$.

Using the Laplace transform I

- Laplace transform of 1st derivative:

$$\begin{aligned}\mathscr{L}\left\{\frac{df(t)}{dt}\right\} &= \int_0^\infty e^{-st}\frac{df(t)}{dt}dt \\ &= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt \\ &= sF(s)-f(0)\end{aligned}$$

- Laplace transform of 2nd derivative:

$$\mathscr{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s)-sf(0)-\frac{df(0)}{dt}$$

- Similarily for the integration:

$$\mathscr{L}\left\{\int_0^t f(u)du\right\} = \frac{F(s)}{s}$$

The Inverse Laplace transform

Definition

The Inverse Laplace transform of a function $f(t)$ is defined by:

$$\mathscr{L}^{-1}\{F(s)\} \equiv F(s) = \frac{1}{2\pi\imath}\int e^{st}F(s)\, ds$$

Are we going to use this definition of the inverse transform to find time function? No! Express $F(s)$ as sum of terms for which we know the inverse transforms!