Discrete Fourier Transform I

We know the inner product of two discrete real functions of length N is:

$$< f , g >= \sum_{n=1}^{N} f(n)\ g(n)$$

(f and g are N dimensional vectors). Now for complex discrete functions of length N, the inner product is:

$$< f , g >= \sum_{n=1}^{N} f(n)\ \overline{g}(n)$$

where $\overline{g}$ is the complex conjugate of g.

Discrete Fourier Transform II

Theorem

The family

$$\left\{ e_k(n) = \exp\left(\frac{\imath 2\pi k n}{N}\right)\right\}_{k=1,\cdots,N}$$

or

$$e_1 = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times 1\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times 1\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 1\times N}{N}\right) \end{pmatrix} \quad e_2 = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times 2\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times 2\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 2\times N}{N}\right) \end{pmatrix} \quad \cdots \quad e_N = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times N\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times N\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times N\times N}{N}\right) \end{pmatrix}$$

is an orthogonal basis of the vectorial space $\mathbb{C}^N$.

Prove this theorem.

Discrete Fourier Transform III

Definition

The discrete Fourier Transform (DFT) of f is:

$$F(k) = < f , e_k >= \sum_{n=1}^{N} f(n)\ \exp\left(\frac{-\imath 2\pi k n}{N}\right)$$

Since $\|e_k\|^2 = N$, the inverse discrete Fourier formula is:

$$f(n) = \frac{1}{N}\sum_{k=1}^{N} F(k)\ \exp\left(\frac{\imath 2\pi k n}{N}\right)$$

Fast Fourier Transform I

A direct calculation of the DFT requires N^2 complex multiplications and additions. The Fast Fourier transform (FFT) reduces the numerical complexity to order $N \log N$ by reorganizing the calculations.

- When the frequency index is even, then we can group the term n and $n+N/2$:

$$F(2k) = \sum_{n=1}^{N/2} (f(n) + f(n+N/2))\ \exp\left(-\frac{\imath 2\pi k n}{N/2}\right)$$

  for even frequencies, the Fourier coefficient is computed by the Fourier transform of a signal $(f(n) + f(n+N/2))$ of length $N/2$

Fast Fourier Transform II

- When the frequency index is odd:

$$F(2k+1) = \sum_{n=1}^{N/2} \exp\left(\frac{\imath 2\pi n}{N}\right)(f(n) - f(n+N/2))\ \exp\left(-\frac{\imath 2\pi k n}{N/2}\right)$$

  which is the Fourier transform of $\exp\left(\frac{\imath 2\pi n}{N}\right)(f(n) - f(n+N/2))$ of length $N/2$.

Two-dimensional Fourier transform

Definition (2-D DFT)

For a 2-dimensional signal, the discrete Fourier transform is defined as:

$$F(k,l) = \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} f(n,m)\ \exp\left(-\frac{\imath 2\pi n k}{N}\right)\ \exp\left(-\frac{\imath 2\pi m l}{M}\right)$$