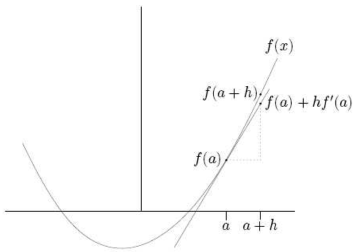
Introduction to the Taylor expansion



We can approximate a point on a curve at $x = a + h$ by the corresponding point on the tangent:

$$f(a+h) \approx f(a) + hf'(a)$$

For h close to 0, it is a good approximation.

Introduction to the Taylor expansion I

- Remember that:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- You can also think about it this way:

$$f'(a) = \frac{f(a+h) - f(a)}{h} + \text{terms which goes to zero as } h \to 0$$

- The terms which goes to zero as $h \to 0$ have to be $h \times$ something and are written $O(h)$. More generally, we note:

Definition:

$$O(h^n) \quad \equiv \quad \text{terms of the form } h^n \times \text{something}$$

Introduction to the Taylor expansion II

- So

$$f'(a) = \frac{f(a+h) - f(a)}{h} + O(h)$$

Multiplying by h, we have :

$$h \cdot f'(a) = f(a+h) - f(a) + O(h^2)$$

or we can write the following Taylor expansion of order 2:

$$f(a+h) = f(a) + h \cdot f'(a) + O(h^2)$$

When h is small (e.g. $h = 0.1$), then h^2 is even smaller ($h^2 = 0.01$).

Introduction to the Taylor expansion III

- Actually this expansion can be done with as many terms as we like:

Taylor's and Maclaurin's Expansions.
For a function f that has n continuous derivatives on the neighbourhood of a, the taylor expansion of f is:

$$f(a+h) = f(a) + f'(a)\ h + f''(a)\ \frac{h^2}{2!} + \cdots + f^{(n)}(a)\ \frac{h^n}{n!} + O(h^{n+1})$$

where $f^{(k)}$ is the $k.th$ derivative of f. When $a = 0$, this expansion is also called the Maclaurin's expansion.

Example: Taylor's Expansion

Example: Compute the first three term of the Taylor expansion of the function $f(x) = e^x$ at $a = 2$.

We have the derivatives $\forall n,\ f^{(n)}(x) = e^x$, so the Taylor expansion becomes:

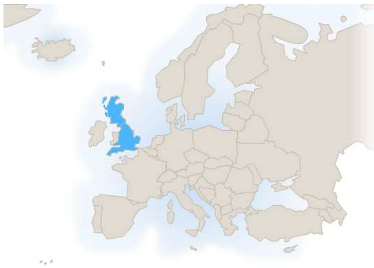
$$\begin{aligned} e^{2+h} &= e^2 + e^2\ h + e^2\ \tfrac{h^2}{2!} + O(h^3) \\ &= e^2 + e^2\ h + e^2\ \tfrac{h^2}{2} + O(h^3) \end{aligned}$$

Exercise: Taylor and Maclaurin expansions

Write down the Maclaurin expansion of:

1. $\sin(x)$
2. $\cos(x)$
3. $\tan(x)$
4. $\sinh(x)$
5. $\cosh(x)$

Brook Taylor and Colin Maclaurin



Brook Taylor
(1685-1731)

Colin Maclaurin
(1698-1746)

Remarks

- The Taylor expansion gives you a finite expansion which approximate the function.
- A Taylor expansion is a finite series approximation of a function around a point.
- Now, we are going to extend this to infinite series.

Taylor series

Taylor's formula with a remainder.
For a function f that has n continuous derivatives on the neighbourhood of a, the taylor expansion of f is:

$$f(x) = f(a) + f'(a)\ (x-a) + f''(a)\ \frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\ \frac{(x-a)^n}{n!} + R_n(x,\xi)$$

with $R_n(x,\xi)$ the remainder term defined as:

$$\begin{aligned} R_n(x,\xi) &= f(x) - \textstyle\sum_{k=0}^{n} f^{(k)}(a)\ \frac{(x-a)^k}{k!} \\ &= f^{(n+1)}(\xi)\ \frac{(x-a)^{n+1}}{(n+1)!} \end{aligned}$$

assuming $f^{(n+1)}(\xi)$ is continuous on $[a;x]$. If $\lim_{n\to\infty} R_n(x,\xi) = 0$, then the function $f(x)$ can be written as an infinite series:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\ \frac{(x-a)^n}{n!}$$

Exercise: Taylor series

Compute the remainder term of the Maclaurin expansion of the following function. Show that $\forall x$, this remainder is converging to 0. Write down then the corresponding Taylor series.

1. e^x
2. $\sin(x)$