To compute the eigenvalues of a square matrix $A$:
1. Compute the matrix $A - \lambda I$.
2. Compute the characteristic equation $\det(A - \lambda I) = 0$.
3. Compute all the eigenvalues as the roots of the characteristic equation.

To compute the eigenvectors for each eigenvalue $\lambda$:
1. Compute the solution $x$ of the linear system $(A - \lambda I)x = 0$.

Exercise:
1. Show that if $x$ is an eigenvector of a matrix $A$ with eigenvalue $\lambda$, then $x' = \alpha x$ is also an eigenvector associated with the same eigenvalue for any real number $\alpha$.
2. Give a geometric interpretation of the previous question.

Definition:
An eigenspace is the set of eigenvectors with a common eigenvalue.

Example:
Consider $u = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ then:
\[
\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & -1 \end{pmatrix} = 1 \times 1 + 0 \times 2 + 2 \times (-1) = -1
\]

Geometric interpretation of eigenvectors and eigenvectors IV

Definition:
The dot product (or inner product or scalar product) of two $n$-component column vectors is:
\[
\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i
\]

Example:
Consider $u = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ the angle is:
\[
\theta = \arccos \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \times \|\mathbf{v}\|} \right)
\]

We can redefine the scalar product:
\[
\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \times \|\mathbf{v}\| \cos(\theta)
\]
Draw in a 3-D space the vectors u and v as defined in the previous slide.

Geometric interpretation of eigenvectors and eigenvalues IX

- A repeated eigenvalue \( \lambda = -2 \). In this case the eigenvectors \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) have to satisfy only one equation: \( x_1 = x_2 - x_3 \). So the set of eigenvectors associated with \( \lambda = -2 \) can be written:

\[
x = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\]

- \( e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) is found by choosing \( x_2 = 0 \) and \( x_1 = 1 \).
- \( e_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \) is found by choosing \( x_1 = 0 \) and \( x_2 = 1 \).
- Warning: when we choose the second eigenvector \( e_2 \), we have to make sure that it is not collinear to the first one \( e_1 \).

Geometric interpretation of eigenvectors and eigenvalues and eigenvectors X

Now, I would like to choose a second eigenvector \( e_2' = (e'_1, e'_2, e'_3) \) such that it is orthogonal to \( e_2 \). \( e_2' \) is an eigenvector or \( A \) associated to the eigenvalue \( \lambda = -2 \). So its components satisfy the equation:

\[
e'_1 = e'_1 - e'_3
\]

- we want \( e_2' \) orthogonal to \( e_2 \), so:

\[
e_2' \cdot e_2 = e'_1 	imes 1 + e'_2 	imes 0 + e'_3 	imes 1 = 0
\]

So we need to solve the following system:

\[
\begin{align*}
e'_1 - e'_3 &= 0 \\
e'_1 + e'_3 &= 0
\end{align*}
\]

which is easily reduced to

\[
\begin{align*}
e'_1 &= -e'_3 \\
e'_3 &= 2e'_1
\end{align*}
\]

Geometric interpretation of eigenvectors and eigenvalues and eigenvectors XI

This is a system with again an infinite number of solutions. So the new eigenvector is expressed as

\[
e_2'' = e_2' \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}
\]

Draw in a 3D space

- the eigenvectors associated to \( \lambda = 4 \) and \( \lambda = -2 \),
- verify that the eigenvectors associated to \( \lambda = 4 \) are not collinear to the ones associated to \( \lambda = -2 \).