Evaluating arithmetic expressions

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Syntax

\[ E \in \text{Exp} ::= n \mid E + E \mid E \times E \]
\[ n \in \text{Nums} ::= 0 \mid 1 \mid 2 \mid \ldots \]

where

- \( n \) ranges over the symbols 0, 1, \ldots representing numbers
- +, \( \times \) \ldots are symbols representing operations

We will always work with abstract syntax
Big-Step Semantics of $\text{Exp}$

**Judgements:**

$E \Downarrow n$

**Meaning:**

Evaluating expressing $E$ should result in numeral $n$.

**Inference rules:**

\[
\frac{}{n \Downarrow n} \quad (\text{B-NUM})
\]

\[
\frac{E_1 \Downarrow n_1 \quad E_2 \Downarrow n_2}{E_1 + E_2 \Downarrow n_3} \quad E_3 = \text{add}(n_1, n_2) \quad (\text{B-ADD})
\]

Similar rules for $\times$, . . .

$\text{add}(\_,-)$ an operation on *numbers* NOT numerals.
Anatomy of a Rule

name

hypothesis . . . hypothesis

conclusion

side-condition

How to Read Axioms

The axiom

\[(B\text{-}NUM)\]

\[n \Downarrow n\]

says:

for every numeral \(n\), it is the case that \(n \Downarrow n\).

In \((B\text{-}NUM)\) \(n\) is a kind of variable: you can put any numeral you like in its place. These are called meta-variables.
How to Read Rules

The inference rule

\[(\text{B-ADD})\]

\[
\frac{E_1 \Downarrow n_1 \quad E_2 \Downarrow n_2}{E_1 + E_2 \Downarrow n_3}
\]

\[n_3 = \text{add}(n_1, n_2)\]

says

for any expressions \(E_1\) and \(E_2\),

if it is the case that \(E_1 \Downarrow n_1\)

and it is the case that \(E_2 \Downarrow n_2\)

then it is the case that \(E_1 + E_2 \Downarrow n_3\)

where \(n_3\) is the numeral such that \(\text{add}(n_1, n_2) = n_3\).

In (B-ADD) \(E_1, E_2, n_1, n_2, n_3\) are meta-variables.

An example derivation

\[
\begin{align*}
3 \Downarrow 3 \quad &\quad \frac{2 \Downarrow 2 \quad (\text{B-NUM})}{2 + 1 \Downarrow 3 \quad (\text{B-ADD})} \quad \frac{1 \Downarrow 1}{(\text{B-NUM})} \\
\end{align*}
\]

This means the judgement

\[3 + (2 + 1) \Downarrow 6\]

can be derived from the rules \((\text{B-NUM}), (\text{B-ADD})\)

Written: \(\vdash_{\text{big}} 3 + (2 + 1) \Downarrow 6\)
Small-step semantics of $Exp$

Judgements:

$$E_1 \rightarrow E_2$$

Meaning:
After performing one step of evaluation of $E_1$ the expression $E_2$ remains to be evaluated.

Inference rules:

(S-LEFT)

$$E_1 \rightarrow E'_1$$

$$E_1 + E_2 \rightarrow E'_1 + E_2$$

(S-N.RIGHT)

$$E_2 \rightarrow E'_2$$

$$n + E_2 \rightarrow n + E'_2$$

(S-ADD)

$$n_1 + n_2 \rightarrow n_3$$

$$n_3 = \text{add}(n_1, n_2)$$

Similar rules for $\times$, . . .

Left-to-right evaluation
Examples

A derivation:

\[
\begin{align*}
3 + 7 & \to 10 & \text{(S-ADD)} \\
(3 + 7) + (8 + 1) & \to 10 + (8 + 1) & \text{(S-LEFT)}
\end{align*}
\]

Another derivation:

\[
\begin{align*}
8 + 1 & \to 9 & \text{(S-ADD)} \\
10 + (8 + 1) & \to 10 + 9 & \text{(S-N.RIGHT)}
\end{align*}
\]

Notation:

\[
\begin{align*}
\vdash_{sm} (3 + 7) + (8 + 1) & \to 10 + (8 + 1) \\
\vdash_{sm} 10 + (8 + 1) & \to 10 + 9
\end{align*}
\]

Choice semantics of \text{Exp}

Inference rules:

\[
\begin{align*}
\frac{E_1 \to_{ch} E_1'}{(\text{S-LEFT})} & \quad E_1 + E_2 \to_{ch} E_1' + E_2 \\
\frac{E_2 \to_{ch} E_2'}{(\text{S-RIGHT})} & \quad E_1 + E_2 \to_{ch} E_1 + E_2' \\
\frac{n_1 + n_2 \to_{ch} n_3}{(\text{S-ADD})} & \quad n_3 = \text{add}(n_1, n_2)
\end{align*}
\]
Examples

A derivation:

\[
\begin{align*}
8 + 1 & \rightarrow_{ch} 9 \\
(3 + 7) + (8 + 1) & \rightarrow_{ch} (3 + 7) + 9
\end{align*}
\]

Notation:

\[
\vdash_{ch} (3 + 7) + (8 + 1) \rightarrow_{ch} (3 + 7) + 9
\]

\[
\vdash_{ch} (3 + 7) + (8 + 1) \rightarrow_{ch} 10 + (8 + 1)
\]

True or False?

\[
\vdash_{ch} E_1 \rightarrow_{ch} E_2 \text{ implies } \vdash_{sm} E_1 \rightarrow E_2
\]

\[
\vdash_{sm} E_1 \rightarrow E_2 \text{ implies } \vdash_{ch} E_1 \rightarrow_{ch} E_2
\]

Executing small-step semantics

The relation \( \rightarrow^k, \ k \in \mathbb{N} \)

We say \( E \rightarrow^k E_k \) holds whenever

\[
E = E_0, \ \vdash_{sm} E_0 \rightarrow E_1, \ \vdash_{sm} E_1 \rightarrow E_2, \ \cdots \vdash_{sm} E_{k-1} \rightarrow E_k
\]

The relation \( \rightarrow^* \)

We say \( E \rightarrow^* F \) holds if \( E \rightarrow^k F \) for some \( k \geq 0 \)

This is called the reflexive transitive closure of \( \rightarrow \).

The final answer

We say that \( n \) is the final answer of \( E \) if \( E \rightarrow^* n \).

Internal consistency of semantics

- Is it possible to derive
  \[ E \downarrow 3 \quad \text{and} \quad E \downarrow 7 \]
  for some expression \( E \)?
- Is there some expression \( E \) which has no resulting value:
  \[ E \rightarrow^* n \quad \text{for no numeral } n \]

Consistency between the different semantics

What is the relationship between the different judgements:

- \( E \downarrow n \)
- \( E \rightarrow^* n \)
- \( E \rightarrow_{ch}^* n \)

Usefulness?
Can these techniques be applied to realistic programming languages?