Real-Reward Testing for Probabilistic Processes

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We introduce a notion of real-valued reward testing for probabilistic processes by extending the traditional nonnegative-reward testing with negative rewards. In this richer testing framework, the may- and must preorders turn out to be inverses. We show that for convergent processes with finitely many states and transitions, but not in the presence of divergence, the real-reward must-testing preorder coincides with the nonnegative-reward must-testing preorder. To prove this coincidence we characterise the usual resolution-based testing in terms of the weak transitions of processes, without having to involve policies, adversaries, schedulers, resolutions or similar structures that are external to the process under investigation. This requires establishing the continuity of our function for calculating testing outcomes.

1 Introduction

Extending classical testing semantics [1, 9] to a setting in which probability and nondeterminism co-exist was initiated in [18]. The application of a test to a process yields a set of probabilities for reaching a success state. Traditionally, this set of result probabilities is obtained by *resolving* [7] a system into a non-empty set of deterministic but probabilistic systems, each representing a possible probabilistic run of the original system; concepts such as *policy* [14], *adversary* [15], *scheduler* [16] and *resolution* [7] have been used for this purpose. *Reward testing* was introduced in [10] for concurrency, though earlier pioneered in [11] for sequential programs; here the success states are labelled by nonnegative real numbers—*rewards*—to indicate degrees of success, and reaching a success state accumulates the associated reward. In [17] an infinite set of success actions is used to report success, and the testing outcomes are vectors of probabilities of performing these success actions. Compared to [10] this amounts to distinguishing different qualities of success, rather than different quantities.

In [18] and [17], both tests and testees are nondeterministic probabilistic processes, whereas [10] allows nonprobabilistic tests only, thereby obtaining a less discriminating form of testing. In [7] we strengthened reward testing by also allowing probabilistic tests. Taking reward testing in this form we showed that for finitary processes, i.e. finite-state and finitely branching processes, all three modes of testing lead to the same testing preorders. Thus, vector-based testing is no more powerful than *scalar* testing that employs only one success action, and likewise reward testing is no more powerful than the special case of reward testing in which all rewards are 1. [1]

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[1] In spite of this there *is* a difference in power between the notions of testing from [18] and [17], but this is an issue that is

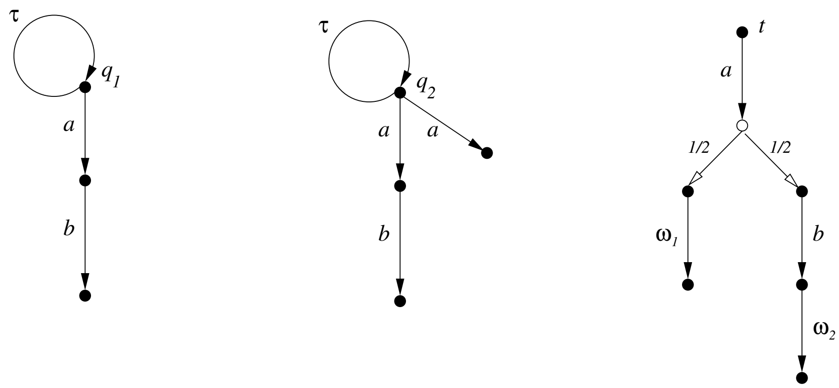

Figure 1: Two processes with divergence and a test

In certain situations it is natural to introduce negative rewards; this is the case, for instance, in the theory of Markov Decision Processes [14]. Intuitively, we could understand negative rewards as costs, while positive rewards are often viewed as benefits or profits. Consider for instance the (nonprobabilistic) processes q_1 and q_2 of Figure 1. Here a represents the action of making an investment. Assuming that the investment is made by bidding for some commodity, the τ-action represents an unsuccessful bid — if this happens one simply tries again. Now b represents the action of reaping the benefits of this investment. Wheres q_1 models a process in which making the investment is always followed by an opportunity to reap the benefits, the process q_2 allows, nondeterministically, for the possibility that the investment is unsuccessful, so that a does not always lead to a state where b is enabled. The test t, which will be explained later, allows us to give a negative reward to action a—its cost—and a positive reward to b.

This leads to the question: *if both negative- and positive rewards are allowed, how would the original reward-testing semantics change?*[2] We refer to the more relaxed form of testing, using positive and negative rewards, as *real-reward testing* and the original one (from [10], but with probabilistic tests as in [7]) as *nonnegative-reward testing*.

The power of real-reward testing is illustrated in Figure 1. The two (nonprobabilistic) processes in the left- and central diagrams are equivalent under (probabilistic) may- as well as must testing; the τ-loops in the initial states cause both processes to fail any nontrivial must test. Yet, if a reward of -1 is associated with performing the action a, and a reward of 2 with the subsequent performance of b, it turns out that in the first process the net reward is either 0, if the process remains stuck in its initial state, or positive, whereas running the second process may yield a loss. See Example 3.8 for details of how these rewards are assigned, and how net rewards are associated with the application of tests such as t. This example shows that for processes that may exhibit divergence, real-reward testing is more discriminating than nonnegative-reward testing, or other forms of probabilistic testing. It also illustrates that the extra power is relevant in applications.

As remarked, in [7] we established that for finitary processes the nonnegative-reward must-testing preorder ($\sqsubseteq_{\mathbf{nr}\mathrm{must}}$) coincides with the probabilistic must-testing preorder ($\sqsubseteq_{\mathrm{pmust}}$), and likewise for the

entirely orthogonal to the distinction between scalar testing, reward testing and vector-based testing. In [17] it is the execution of a success *action* that constitutes success, whereas in [1, 9, 18, 10] it is reaching a success *state* (even though typically success actions are used to identify those states). In [2, Ex 5.3] we showed that state-based testing is (slightly) more powerful than action-based testing. The results presented in [7] about the coincidence of scalar, reward, and vector-based testing preorders pertain to action-based version of each, but in the conclusion it is observed that the same coincidence could be obtained for their state-based versions. In the current paper we stick to state-based testing.

[2]One might suspect no change at all, for any assignment of rewards from the interval $[-1,+1]$ can be converted into a non-negative assignment simply by adding 1 to all of them. But that would not preserve the testing order in the case of zero-outcomes that resulted from a process's failing to reach any success state at all: those zeroes would remain zero.

$$(\sqsubseteq_{\mathbf{rr}\text{may}})^{-1} \overset{\text{Thm. 3.7}}{=} \sqsubseteq_{\mathbf{rr}\text{must}} \overset{\text{Thm. 6.4}}{=} \sqsubseteq_{\mathbf{nr}\text{must}} \overset{[7]}{=} \sqsubseteq_{\text{pmust}} \overset{[3]}{=} \sqsubseteq_{FS}$$

The symbol $=$ between two relations means that they coincide for finitary convergent processes.

Figure 2: The relationship of different testing preorders.

may preorders. The main result of this paper is that restricted to finitary convergent processes, the real-reward must preorder $\sqsubseteq_{\mathbf{rr}\text{must}}$ coincides with the nonnegative-reward must preorder, i.e. for any finitary convergent processes,

$$\Delta \sqsubseteq_{\mathbf{rr}\text{must}} \Gamma \quad \text{iff} \quad \Delta \sqsubseteq_{\mathbf{nr}\text{must}} \Gamma. \tag{1}$$

Here, as we shall see, convergence is the natural generalisation of the standard concept for nonprobabilistic processes to the probabilistic setting; in particular it rules out the processes of Figure 1.

There is also a surprisingly simple proof of the fact that for real-reward testing the may- and must preorders are the inverse of each other, i.e. that for any processes Δ and Γ,

$$\Delta \sqsubseteq_{\mathbf{rr}\text{may}} \Gamma \quad \text{iff} \quad \Gamma \sqsubseteq_{\mathbf{rr}\text{must}} \Delta. \tag{2}$$

This pleasing symmetry does not hold for the more restrictive nonnegative-reward (or scalar) testing. Moreover, the analogy of (1) for the may preorder does not hold, i.e. $\sqsubseteq_{\mathbf{rr}\text{may}}$ does not coincide with $\sqsubseteq_{\mathbf{nr}\text{may}}$ (q.v. the end of Section 8).

Although it is easy to see that in (1) the former implies the latter, to prove the opposite is far from trivial; see more discussion in Section 7. We employ a characterisation of $\sqsubseteq_{\text{pmust}}$ from [2, 3]. *Failure simulation* is a well-known behavioural preorder for nondeterministic processes [8]; in [2] we showed that it could be adapted to characterise the probabilistic must-testing preorder $\sqsubseteq_{\text{pmust}}$, and in [3] this work was generalised from finite to finitary processes. This involved the generalisation of the standard notion of (weak) derivations in state-based systems [13], to probabilistic processes, i.e. probability distributions. By capitalising on this novel notion of derivation between distributions we can show that the failure simulation preorder $\sqsubseteq_{FS}$ is contained in $\sqsubseteq_{\mathbf{rr}\text{must}}$. Convergence is essential here, even though it is not needed to establish that $\sqsubseteq_{FS}$ is contained in $\sqsubseteq_{\mathbf{nr}\text{must}}$. Recall that $\sqsubseteq_{\mathbf{rr}\text{must}}$ is defined using *resolutions*; the key to proving this containment, the heart of the paper, is showing that certain derivations, which we call *extreme derivations*, are essentially the same as *resolutions*. Combining this with the results from [7] and [3] mentioned above leads to our required result that $\sqsubseteq_{\mathbf{nr}\text{must}}$ is included in $\sqsubseteq_{\mathbf{rr}\text{must}}$, as far as finitary convergent processes are concerned. Consequently, in this case, all the relations of Figure 2 collapse into one.

The rest of this paper is organised as follows. We start by recalling notation for probabilistic labelled transition systems. In Section 3 we review the resolution-based testing approach and show that the real-reward may preorder is simply the inverse of the real-reward must preorder. Moreover, using the example of Figure 1, we show that in the presence of divergence the inclusion of $\sqsubseteq_{\mathbf{rr}\text{must}}$ in $\sqsubseteq_{\mathbf{nr}\text{must}}$ is proper. In Section 4 we recall the notions of derivation and the failure simulation preorder. In Section 5 we show that resolutions can be seen as certain kinds of derivations. Then in Section 6 we show for finitary convergent processes that real-reward must testing coincides with nonnegative-reward must testing. We explain in Section 7 why the proof of the coincidence result cannot easily be simplified, and then conclude in Section 8.

Besides the related work already mentioned above, many other studies on probabilistic testing and simulation semantics have appeared in the literature. They are reviewed in [6, 2]. An extended abstract of the current work has appeared as [5]. All the proofs omitted there are now detailed. Section 7 is newly added to explain the subtle difference between $\sqsubseteq_{\mathbf{rr}\text{must}}$ and $\sqsubseteq_{\mathbf{nr}\text{must}}$.

2 Probabilistic Processes

A (discrete) probability *subdistribution* over a set S is a function $\Delta : S \to [0,1]$ with $\sum_{s\in S}\Delta(s) \leq 1$; the *support* of such a Δ is $\lceil\Delta\rceil := \{\, s\in S \mid \Delta(s) > 0 \,\}$, and its *mass* $|\Delta|$ is $\sum_{s\in\lceil\Delta\rceil}\Delta(s)$. A subdistribution is a (total, or full) *distribution* if $|\Delta| = 1$. The point distribution $\overline{s}$ assigns probability 1 to s and 0 to all other elements of S, so that $\lceil\overline{s}\rceil = \{s\}$. With $\mathscr{D}_{sub}(S)$ we denote the set of subdistributions over S, and with $\mathscr{D}(S)$ its subset of full distributions.

Let $\{\Delta_k \mid k \in K\}$ be a set of subdistributions, possibly infinite. Then $\sum_{k\in K}\Delta_k$ is the real-valued function in $S \to \mathbb{R}$ defined by $(\sum_{k\in K}\Delta_k)(s) := \sum_{k\in K}\Delta_k(s)$. This is a partial operation on subdistributions because for some state s the sum of $\Delta_k(s)$ might exceed 1. If the index set is finite, say $\{1..n\}$, we often write $\Delta_1 + \ldots + \Delta_n$. For p a real number from $[0,1]$ we use $p\cdot\Delta$ to denote the subdistribution given by $(p\cdot\Delta)(s) := p\cdot\Delta(s)$. Finally we use ε to denote the everywhere-zero subdistribution that thus has empty support. These operations on subdistributions do not readily adapt themselves to distributions; yet if $\sum_{k\in K} p_k = 1$ for some $p_k \geq 0$, and the Δ_k are distributions, then so is $\sum_{k\in K} p_k\cdot\Delta_k$.

The expected value $\sum_{s\in S}\Delta(s)\cdot f(s)$ over a subdistribution Δ of a bounded nonnegative function f to the reals or tuples of them is written $\mathrm{Exp}_\Delta(f)$, and the image of a subdistribution Δ through a function $f : S \to T$, for some set T, is written $\mathrm{Img}_f(\Delta)$ — the latter is the subdistribution over T given by $\mathrm{Img}_f(\Delta)(t) := \sum_{f(s)=t}\Delta(s)$ for each $t \in T$.

Definition 2.1 A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, \mathsf{Act}, \to\rangle$, where
(i) S is a set of states,
(ii) Act is a set of visible actions,
(iii) relation $\to$ is a subset of $S \times \mathsf{Act}_\tau \times \mathscr{D}(S)$.

Here Act_τ denotes $\mathsf{Act} \cup \{\tau\}$, where $\tau \notin \mathsf{Act}$ is the invisible- or internal action.

A (nonprobabilistic) labelled transition system (LTS) may be viewed as a degenerate pLTS — one in which only point distributions are used. As with LTSs, we write $s \xrightarrow{\alpha} \Delta$ for $(s,\alpha,\Delta)\in\to$, as well as $s \xrightarrow{\alpha}$ for $\exists\Delta : s \xrightarrow{\alpha} \Delta$ and $s \to$ for $\exists\alpha : s \xrightarrow{\alpha}$, with $s \stackrel{\alpha}{\not\to}$ and $s \not\to$ representing their negations.

We graphically depict pLTSs as follows. States are represented by nodes of the form $\bullet$ and distributions by nodes of the form $\circ$. For any state s and distribution Δ with $s \xrightarrow{\alpha} \Delta$ we draw an edge from s to Δ, labelled with α. For any distribution Δ and state s in $\lceil\Delta\rceil$, the support of Δ, we draw an edge from Δ to s, labelled with $\Delta(s)$. We leave out point-distributions, diverting an incoming edge to the unique state in its support. See e.g. Figure 4 in the next section for some example pLTSs.

In this paper a *(probabilistic) process* will simply be a distribution over the state set of a pLTS. A pLTS is *deterministic* if for any state s and label α there is at most one distribution Δ with $s \xrightarrow{\alpha} \Delta$. It is *finitely branching* if the set $\{\Delta \mid s \xrightarrow{\alpha} \Delta,\ \alpha\in L\}$ is finite for all states s; if moreover S is finite, then the pLTS is *finitary*. A subdistribution Δ over the state set S of an arbitrary pLTS is *finitary* if restricting S to the states reachable from Δ in the graphical representation of the pLTS yields a finitary sub-pLTS. Similarly, a subdistribution Δ is *finite* if restricting S to the states reachable from Δ yields a finitary sub-pLTS without loops.

3 Testing probabilistic processes

A *test* is a finite distribution over the state set of a pLTS having $\mathsf{Act}_\tau \cup \Omega$ as its set of transition labels, where Ω is a set of fresh *success* actions, not already in Act_τ, introduced specifically to report testing outcomes.[3] For simplicity we may assume a fixed pLTS of processes—our results apply to any choice

[3]For *vector-based* testing we normally take Ω to be countably infinite [17]. This way we have an unbounded supply of success actions for building tests, of course without obligation to use them all. *Scalar* testing is obtained by taking $|\Omega| = 1$.

$$\frac{t \xrightarrow{\alpha}_{\mathbf{T}} \Theta \qquad \alpha \notin \mathsf{Act}}{t \| p \xrightarrow{\alpha} \Theta \| \overline{p}} \qquad \frac{p \xrightarrow{\tau}_{\mathbf{P}} \Delta}{t \| p \xrightarrow{\tau} \overline{t} \| \Delta} \qquad \frac{t \xrightarrow{a}_{\mathbf{T}} \Theta \quad p \xrightarrow{a}_{\mathbf{P}} \Delta \quad a \in \mathsf{Act}}{t \| p \xrightarrow{\tau} \Theta \| \Delta}$$

Figure 3: Synchronous parallel composition between tests and processes

of such a pLTS—and a fixed pLTS of tests. Since the power of testing depends on the expressivity of the pLTS of tests—in particular certain types of tests are necessary for our results—let us just postulate that this pLTS is sufficiently expressive for our purposes — for example that it can be used to interpret all processes from the language pCSP, as in our previous papers [6, 2, 3]. [4]

Although we use success *actions*, they are used merely to mark certain states as success states, namely the sources of transitions labelled by success actions. For this reason we systematically ignore the distributions that can be reached after a success action. We impose two requirements on all states in a pLTS of tests, namely

(A) if $t \xrightarrow{\omega_1}$ and $t \xrightarrow{\omega_2}$ with $\omega_1, \omega_2 \in \Omega$ then $\omega_1 = \omega_2$. *uniqueness*

(B) if $t \xrightarrow{\omega}$ with $\omega \in \Omega$ and $t \xrightarrow{\alpha} \Delta$ with $\alpha \in \mathsf{Act}_\tau$ then $u \xrightarrow{\omega}$ for all $u \in \lceil \Delta \rceil$. *no ω-disabling*

The first condition says that a success state can have one success identity only, whereas the second condition is a slight weakening of the requirement from [10] that success states must be end states; it allows further progress from an ω-success state, for some $\omega \in \Omega$, but ω must remain enabled. [5]

To apply test Θ to process Δ we form a parallel composition $\Theta \| \Delta$ in which *all* visible actions of Δ must synchronise with Θ. Those synchronisations are immediately renamed into τ so that the resulting composition is a process whose only possible actions are the elements of $\Omega_\tau := \Omega \cup \{\tau\}$. Formally, if $\langle \mathbf{P}, \mathsf{Act}, \rightarrow_{\mathbf{P}} \rangle$ and $\langle \mathbf{T}, \mathsf{Act} \cup \Omega, \rightarrow_{\mathbf{T}} \rangle$ are the pLTSs of processes and tests, then the pLTS of applications of tests to processes is $\langle \mathbf{C}, \Omega, \rightarrow \rangle$, with $\mathbf{C} = \{t \| p \mid t \in \mathbf{T} \wedge p \in \mathbf{P}\}$ and $\rightarrow$ the transition relation generated by the rules in Fig. 3. Here if $\Theta \in \mathscr{D}(\mathbf{T})$ and $\Delta \in \mathscr{D}(\mathbf{P})$, then $\Theta \| \Delta$ is the distribution given by $(\Theta \| \Delta)(t \| p) := \Theta(t) \cdot \Delta(p)$. The resulting pLTS also satisfies (A), (B) above; this would not be the case if we had strengthened (B) to require that success states must be end states.

We will define the result $\mathscr{A}(\Theta, \Delta)$ of applying the test Θ to the process Δ to be a set of testing outcomes, exactly one of which results from each resolution of the choices in $\Theta \| \Delta$. Each *testing outcome* is an Ω-tuple of real numbers in the interval [0,1], i.e. a function $o : \Omega \rightarrow [0,1]$, and its ω-component $o(\omega)$, for $\omega \in \Omega$, gives the probability that the resolution in question will reach an ω-*success state*, one in which the success action ω is possible.

Due to the presence of nondeterminism in pLTSs, we need a mechanism to reduce a nondeterministic structure into a set of deterministic structures, each of which determines a single possible outcome. Here we adapt the notion of *resolution*, defined in [7] for probabilistic automata, to pLTSs.

Definition 3.1 [Resolution] A *resolution* of a subdistribution $\Phi \in \mathscr{D}_{sub}(S)$ in a pLTS $\langle S, \Omega, \rightarrow \rangle$ is a triple $\langle R, \Lambda, \rightarrow_R \rangle$ where $\langle R, \Omega, \rightarrow_R \rangle$ is a deterministic pLTS and $\Lambda \in \mathscr{D}_{sub}(R)$, such that there exists a *resolving function* $f : R \rightarrow S$ satisfying

(i) $\mathrm{Img}_f(\Lambda) = \Phi$

(ii) if $r \xrightarrow{\alpha}_R \Lambda'$ for $\alpha \in \Omega_\tau$ then $f(r) \xrightarrow{\alpha} \mathrm{Img}_f(\Lambda')$

(iii) if $f(r) \xrightarrow{\alpha}$ for $\alpha \in \Omega_\tau$ then $r \xrightarrow{\alpha}_R$.

[4] In [3] tests are allowed to be finitary, but if two processes are behaviourally different they can be distinguished by some characteristic tests which are always finite. Therefore, the results in [3] still hold if tests are required to be finite, as we do here.

[5] This simplifies our treatment of test but, as can be seen from Appendix A of [7], it is not a heavy restriction.

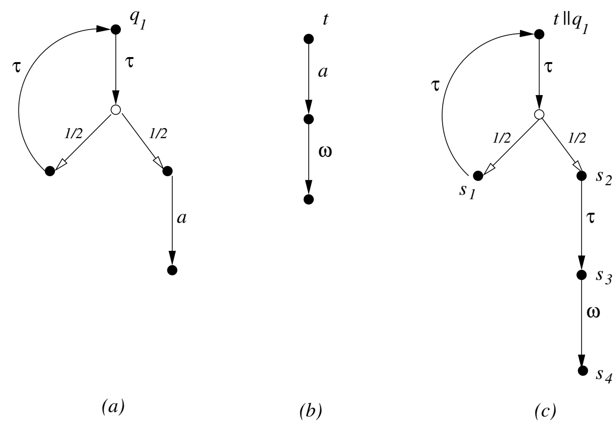

Figure 4: Testing the process $\overline{q_1}$

The reader is referred to Section 2 of [7] for a detailed discussion of the concept of resolution, and the manner in which a resolution represents a run of a process; in particular in a resolution states in S are allowed to be resolved into distributions, and computation steps can be *probabilistically interpolated*. Our resolutions match the results of applying a scheduler as defined in [16].

We now explain how to associate an outcome with a particular resolution, which in turn will associate a set of outcomes with a subdistribution in a pLTS. Given a deterministic pLTS $\langle R, \Omega, \rightarrow_R \rangle$ consider the functional $\mathscr{F} : (R \rightarrow [0,1]^\Omega) \rightarrow (R \rightarrow [0,1]^\Omega)$ defined by

$$\mathscr{F}(g)(r)(\omega) := \begin{cases} 1 & \text{if } r \xrightarrow{\omega} \\ 0 & \text{if } r \not\xrightarrow{\omega} \text{ and } r \not\xrightarrow{\tau} \\ \mathrm{Exp}_\Delta(g)(\omega) & \text{if } r \not\xrightarrow{\omega} \text{ and } r \xrightarrow{\tau} \Delta. \end{cases} \tag{3}$$

We view the unit interval $[0,1]$ ordered in the standard manner as a complete lattice; this induces the structure of a complete lattice on the product $[0,1]^\Omega$ and in turn on the set of functions $R \rightarrow [0,1]^\Omega$. The functional $\mathscr{F}$ is easily seen to be monotonic and therefore has a least fixed point, which we denote by $\mathbb{V}_{\langle R,\Omega,\rightarrow_R \rangle}$; this is abbreviated to $\mathbb{V}$ when the deterministic pLTS in question is understood. Intuitively $\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R \rangle})$ is the result of executing the resolution $\langle R, \Lambda, \rightarrow_R \rangle$ starting from the initial distribution Λ, a vector of probabilities. From Definition 3.1 we see that in general a distribution Φ gives rise to a non-empty set of resolutions. Collecting all of the possible results of executing them we get

$$\mathscr{A}(\Phi) = \{ \mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R \rangle}) \mid \langle R, \Lambda, \rightarrow_R \rangle \text{ is a resolution of } \Phi \} \,. \tag{4}$$

This notation is most often used in calculating the results of applying a test to a process. To emphasise this, we will sometimes use the notation $\mathscr{A}(\Theta, \Delta)$ for $\mathscr{A}(\Theta \| \Delta)$.

Example 3.2 Consider the process $\overline{q_1}$ depicted in Figure 4(a). When we apply the test $\bar{t}$ depicted in Figure 4(b) to it we get the process $\overline{t \| q_1}$ depicted in Figure 4(c). This process is already deterministic, hence has essentially only one resolution: itself. Moreover the outcome $\mathrm{Exp}_{\overline{t \| q_1}}(\mathbb{V}) = \mathbb{V}(t \| q_1)$ associated with it is the least solution of the equation $\mathbb{V}(t \| q_1) = \frac{1}{2} \cdot \mathbb{V}(t \| q_1) + \frac{1}{2} \overrightarrow{\omega}$ where $\overrightarrow{\omega} : \Omega \rightarrow [0,1]$ is the Ω-tuple with $\overrightarrow{\omega}(\omega) = 1$ and $\overrightarrow{\omega}(\omega') = 0$ for all $\omega' \neq \omega$. In fact this equation has a unique solution in $[0,1]^\Omega$, namely $\overrightarrow{\omega}$. Thus $\mathscr{A}(\bar{t}, \overline{q_1}) = \{\overrightarrow{\omega}\}$. $\square$

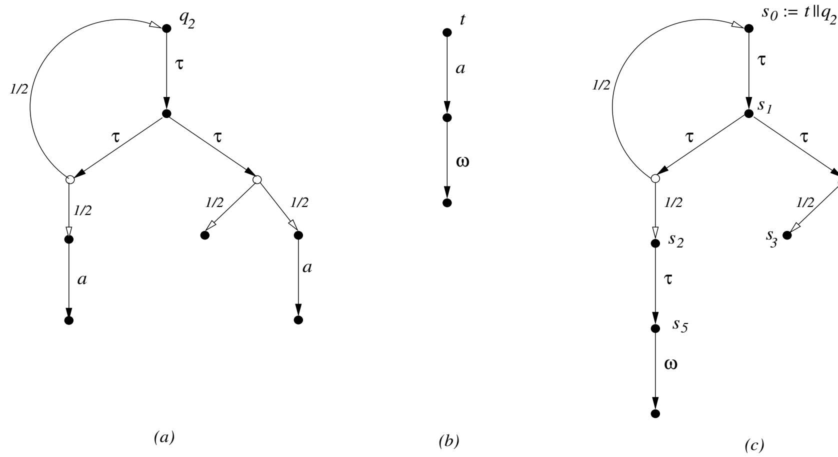

Figure 5: Testing the process $\overline{q_2}$

Example 3.3 Consider the process $\overline{q_2}$ and the application of the test $\bar{t}$ to it, as outlined in Figure 5. For each $k \geq 1$ the process $\overline{t \| q_2}$ has a resolution $\langle R_k, \Lambda, \rightarrow_{R_k} \rangle$ such that $\mathrm{Exp}_\Lambda(\mathbb{V}) = (1 - \frac{1}{2^k})\overrightarrow{\omega}$; intuitively it goes around the loop $(k-1)$ times before at last taking the right hand τ action. Thus $\mathscr{A}(\bar{t}, \overline{q_2})$ contains $(1 - \frac{1}{2^k})\overrightarrow{\omega}$ for every $k \geq 1$. But it also contains $\overrightarrow{\omega}$, because of the resolution which takes the left hand τ-move every time. Thus $\mathscr{A}(\bar{t}, \overline{q_2})$ includes the set

$$\{(1 - \tfrac{1}{2})\overrightarrow{\omega},\ (1 - \tfrac{1}{2^2})\overrightarrow{\omega}, \ldots, (1 - \tfrac{1}{2^k})\overrightarrow{\omega}, \ldots, \overrightarrow{\omega}\}$$

As resolutions allow any interpolation between the two τ-transitions from state s_1, $\mathscr{A}(\bar{t}, \overline{q_2})$ is actually the convex closure of the above set. □

There are two standard methods for comparing two sets of ordered outcomes:

$$\begin{array}{ll} O_1 \leq_{\mathrm{Ho}} O_2 & \text{if for every } o_1 \in O_1 \text{ there exists some } o_2 \in O_2 \text{ such that } o_1 \leq o_2 \\ O_1 \leq_{\mathrm{Sm}} O_2 & \text{if for every } o_2 \in O_2 \text{ there exists some } o_1 \in O_1 \text{ such that } o_1 \leq o_2 \end{array}$$

This gives us our definition of the probabilistic may- and must-testing preorders; they are decorated with $\cdot^\Omega$ for the repertoire Ω of testing actions they employ.

Definition 3.4 [Probabilistic testing preorders]

(i) $\Delta \sqsubseteq^\Omega_{\mathrm{pmay}} \Gamma$ if for every Ω-test Θ, $\mathscr{A}(\Theta, \Delta) \leq_{\mathrm{Ho}} \mathscr{A}(\Theta, \Gamma)$.

(ii) $\Delta \sqsubseteq^\Omega_{\mathrm{pmust}} \Gamma$ if for every Ω-test Θ, $\mathscr{A}(\Theta, \Delta) \leq_{\mathrm{Sm}} \mathscr{A}(\Theta, \Gamma)$.

These preorders are abbreviated to $\Delta \sqsubseteq_{\mathrm{pmay}} \Gamma$ and $\Delta \sqsubseteq_{\mathrm{pmust}} \Gamma$ when $|\Omega| = 1$.

In [7] we established that for finitary processes $\sqsubseteq^\Omega_{\mathrm{pmay}}$ coincides with $\sqsubseteq_{\mathrm{pmay}}$ and $\sqsubseteq^\Omega_{\mathrm{pmust}}$ with $\sqsubseteq_{\mathrm{pmust}}$ for any choice of Ω. We also defined the reward-testing preorders in terms of the mechanism set up so far. The idea is to associate with each success action $\omega \in \Omega$ a reward, which is a nonnegative number in the unit interval $[0,1]$; and then a run of a probabilistic process in parallel with a test yields an expected reward accumulated by those states which can enable success actions. A reward tuple $h \in [0,1]^\Omega$ is used to assign reward $h(\omega)$ to success action ω, for each $\omega \in \Omega$. Due to the presence of nondeterminism, the application of a test Θ to a process Δ produces a set of expected rewards. Two sets of rewards can be compared by examining their suprema/infima; this gives us two methods of testing called reward may/must testing. In [7] all rewards are required to be nonnegative, so we refer to that approach of testing as *nonnegative-reward testing*. If we also allow negative rewards, which intuitively can be understood as costs, then we obtain an approach of testing called *real-reward testing*. Technically, we simply let reward tuples h range over the set $[-1,1]^\Omega$. If $o \in [0,1]^\Omega$, we use the dot-product $h \cdot o = \sum_{\omega\in\Omega} h(\omega)\cdot o(\omega)$. It can apply to a set $O \subseteq [0,1]^\Omega$ so that $h \cdot O = \{h \cdot o \mid o \in O\}$. Let $A \subseteq [-1,1]$. We use the notation $\bigsqcup A$ for the supremum of set A, and $\bigsqcap A$ for the infimum.

Definition 3.5 [Reward testing preorders]

(i) $\Delta \sqsubseteq^\Omega_{\mathbf{nr}\mathrm{may}} \Gamma$ if for every Ω-test Θ and nonnegative-reward tuple $h \in [0,1]^\Omega$, $\bigsqcup h \cdot \mathscr{A}(\Theta,\Delta) \le \bigsqcup h \cdot \mathscr{A}(\Theta,\Gamma)$.

(ii) $\Delta \sqsubseteq^\Omega_{\mathbf{nr}\mathrm{must}} \Gamma$ if for every Ω-test Θ and nonnegative-reward tuple $h \in [0,1]^\Omega$, $\bigsqcap h \cdot \mathscr{A}(\Theta,\Delta) \le \bigsqcap h \cdot \mathscr{A}(\Theta,\Gamma)$.

(iii) $\Delta \sqsubseteq^\Omega_{\mathbf{rr}\mathrm{may}} \Gamma$ if for every Ω-test Θ and real-reward tuple $h \in [-1,1]^\Omega$, $\bigsqcup h \cdot \mathscr{A}(\Theta,\Delta) \le \bigsqcup h \cdot \mathscr{A}(\Theta,\Gamma)$.

(iv) $\Delta \sqsubseteq^\Omega_{\mathbf{rr}\mathrm{must}} \Gamma$ if for every Ω-test Θ and real-reward tuple $h \in [-1,1]^\Omega$, $\bigsqcap h \cdot \mathscr{A}(\Theta,\Delta) \le \bigsqcap h \cdot \mathscr{A}(\Theta,\Gamma)$.

This time we drop the superscript Ω iff Ω is countably infinite.

It is shown in Corollary 1 of [7] that nonnegative-reward testing is equally powerful as probabilistic testing.

Theorem 3.6 [7] For any finitary processes Δ and Γ,

(i) $\Delta \sqsubseteq_{\mathbf{nr}\mathrm{may}} \Gamma$ if and only if $\Delta \sqsubseteq_{\mathrm{pmay}} \Gamma$.

(ii) $\Delta \sqsubseteq_{\mathbf{nr}\mathrm{must}} \Gamma$ if and only if $\Delta \sqsubseteq_{\mathrm{pmust}} \Gamma$.

In this paper we focus on the real-reward testing preorders $\sqsubseteq_{\mathbf{rr}\mathrm{may}}$ and $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$, by comparing them with the nonnegative reward testing preorders $\sqsubseteq_{\mathbf{nr}\mathrm{may}}$ and $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$. Although these two nonnegative-reward testing preorders are in general incomparable, we have for the real-reward testing preorders:

Theorem 3.7 For any processes Δ and Γ, it holds that $\Delta \sqsubseteq_{\mathbf{rr}\mathrm{may}} \Gamma$ if and only if $\Gamma \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Delta$.

Proof: We first notice that for any nonempty set $A \subseteq [0,1]^\Omega$ and any reward tuple $h \in [-1,1]^\Omega$,

$$\bigsqcup h \cdot A \quad = \quad -(\bigsqcap (-h) \cdot A) \tag{5}$$

where $-h$ is the negation of h, i.e. $(-h)(\omega) = -(h(\omega))$ for any $\omega \in \Omega$. We consider the "if" direction; the "only if" direction is similar. Let Θ be any Ω-test and h be any real reward tuple in $[-1,1]^\Omega$. Clearly, $-h$ is also a real reward tuple. Suppose $\Gamma \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Delta$, then

$$\bigsqcap (-h) \cdot \mathscr{A}(\Theta,\Gamma) \quad \le \quad \bigsqcap (-h) \cdot \mathscr{A}(\Theta,\Delta) \tag{6}$$

Therefore, we can infer that

$$\begin{array}{rcll} \bigsqcup h \cdot \mathscr{A}(\Theta,\Delta) & = & -(\bigsqcap (-h) \cdot \mathscr{A}(\Theta,\Delta)) & \text{by (5)} \\ & \le & -(\bigsqcap (-h) \cdot \mathscr{A}(\Theta,\Gamma)) & \text{by (6)} \\ & = & \bigsqcup h \cdot \mathscr{A}(\Theta,\Gamma) & \text{by (5).} \end{array}$$

□

Our next task is to compare $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$ with $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$. The former is included in the latter, which directly follows from Definition 3.5. Surprisingly, it turns out that for finitary convergent processes the latter is also included in the former, thus establishing that the two preorders are in fact the same. The rest of the paper is devoted to proving this result. However, we first show that this result does not extend to divergent processes.

Example 3.8 Consider the processes $\overline{q_1}$ and $\overline{q_2}$ depicted in Figure 1. Using the characterisations of $\sqsubseteq_{\mathrm{pmay}}$ and $\sqsubseteq_{\mathrm{pmust}}$ in [3], it is easy to see that these processes cannot be distinguished by probabilistic may- and must testing, and hence not by nonnegative-reward testing either. However, let $\bar{t}$ be the test in the right diagram of Figure 1 that first synchronises on the action a, and then with probability $\frac{1}{2}$ reaches a state in which a reward of -2 is allocated, and with the remaining probability $\frac{1}{2}$ synchronises with the action b and reaches a state that yields a reward of 4. Thus the test employs two success actions ω_1 and ω_2, and we use the reward tuple h with $h(\omega_1) = -2$ and $h(\omega_2) = 4$. Then the resolution of $\overline{q_1}$ that does not involve the τ-loop contributes the value $-2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = -1 + 2 = 1$ to the set $h \cdot \mathscr{A}(\bar{t}, \overline{q_1})$, whereas the resolution that only involves the τ-loop contributes the value 0. Due to interpolation, $h \cdot \mathscr{A}(\bar{t}, \overline{q_1})$ is in fact the entire interval $[0,1]$. On the other hand, the resolution corresponding to the a-branch of q_2 contributes the value -1 and $h \cdot \mathscr{A}(\bar{t}, \overline{q_2}) = [-1,1]$. Thus $\bigsqcap h \cdot \mathscr{A}(\bar{t}, \overline{q_1}) = 0 > -1 = \bigsqcap h \cdot \mathscr{A}(\bar{t}, \overline{q_2})$, and hence $\overline{q_1} \not\sqsubseteq_{\mathbf{rr}\mathrm{must}} \overline{q_2}$. $\Box$

4 Failure simulations

In this section we explain the characterisation of probabilistic testing from [2, 3]; it depends on a generalisation of failure simulations [8] to the probabilistic setting. The key ingredient is that of weak derivations for distributions. To deal with infinite (but finitary) processes, we need to employ the weak derivations of [3] rather than those of [2].

In a pLTS actions are performed only by states, in that actions are given by relations from states to distributions. But processes in general correspond to distributions over states, so in order to define what it means for a process to perform an action, we need to *lift* these relations so that they also apply to distributions. In fact we will find it convenient to lift them to subdistributions.

Definition 4.1 Let $(S, L, \rightarrow)$ be a pLTS and $\mathscr{R} \subseteq S \times \mathscr{D}_{sub}(S)$ be a relation from states to subdistributions. Then $\overline{\mathscr{R}} \subseteq \mathscr{D}_{sub}(S) \times \mathscr{D}_{sub}(S)$ is the smallest relation that satisfies:
(i) $s \,\mathscr{R}\, \Delta$ implies $\bar{s} \,\overline{\mathscr{R}}\, \Delta$, and
(ii) (Linearity) $\Gamma_i \,\overline{\mathscr{R}}\, \Delta_i$ for $i \in I$ implies $(\sum_{i\in I} p_i \cdot \Gamma_i) \;\overline{\mathscr{R}}\; (\sum_{i\in I} p_i \cdot \Delta_i)$ for any $p_i \in [0,1]$ $(i \in I)$ with $\sum_{i\in I} p_i \leq 1$, where I is a countable set.

An application of this notion is when the relation is $\xrightarrow{\alpha}$ for $\alpha \in \mathsf{Act}_\tau$; in that case we also write $\xrightarrow{\alpha}$ for $\overline{\xrightarrow{\alpha}}$. Thus, as source of a relation $\xrightarrow{\alpha}$ we now also allow distributions, and even subdistributions. A subtlety of this approach is that for any action α, we have $\varepsilon \xrightarrow{\alpha} \varepsilon$ simply by taking $I = \emptyset$ or $\sum_{i\in I} p_i = 0$ in Definition 4.1. That turns out to make ε especially useful for modelling the "chaotic" aspects of divergence in [3], in particular that in the must-case a divergent process can mimic any other.

Definition 4.1 is very similar to our previous definition in [2], although there it applied only to (full) distributions:

Lemma 4.2 $\Gamma \,\overline{\mathscr{R}}\, \Delta$ if and only if
(i) $\Gamma = \sum_{i\in I} p_i \cdot \overline{s_i}$, where I is an index set and $\sum_{i\in I} p_i \leq 1$,
(ii) For each $i \in I$ there is a subdistribution Δ_i such that $s_i \,\mathscr{R}\, \Delta_i$,
(iii) $\Delta = \sum_{i\in I} p_i \cdot \Delta_i$.

Proof: Straightforward. □

An important point here is that a single state can be split into several pieces: that is, the decomposition of Γ into $\sum_{i\in I} p_i \cdot \overline{s_i}$ is not unique.

Definition 4.3 [Weak derivation] Suppose we have subdistributions $\Delta, \Delta_k^{\rightarrow}, \Delta_k^{\times}$, for $k \geq 0$, with the following properties:

$$\begin{array}{rcl} \Delta & = & \Delta_0^{\rightarrow} + \Delta_0^{\times} \\ \Delta_0^{\rightarrow} & \xrightarrow{\tau} & \Delta_1^{\rightarrow} + \Delta_1^{\times} \\ & \vdots & \\ \Delta_k^{\rightarrow} & \xrightarrow{\tau} & \Delta_{k+1}^{\rightarrow} + \Delta_{k+1}^{\times} \ . \\ & \vdots & \end{array}$$

Then we call $\Delta' := \sum_{k=0}^{\infty} \Delta_k^{\times}$ a *weak derivative* of Δ, and write $\Delta \Longrightarrow \Delta'$ to mean that Δ can make a *weak derivation* to its derivative Δ'.

There is always at least one weak derivative of any subdistribution (the subdistribution itself) and there can be many.

Proposition 4.4 [Transitivity, linearity and decomposition of weak derivations [4]]

(i) If $\Delta \Longrightarrow \Delta'$ and $\Delta' \Longrightarrow \Delta''$ then $\Delta \Longrightarrow \Delta''$.

Let $p_i \in [0,1]$ for $i \in I$ with $\sum_{i\in I} p_i \leq 1$.

(ii) If $\Delta_i \Longrightarrow \Delta_i'$ for all $i \in I$ then $\sum_{i\in I} p_i \cdot \Delta_i \Longrightarrow \sum_{i\in I} p_i \cdot \Delta_i'$.

(iii) If $\sum_{i\in I} p_i \cdot \Delta_i \Longrightarrow \Delta'$ then $\Delta' = \sum_{i\in I} p_i \cdot \Delta_i'$ for subdistributions Δ_i' such that $\Delta_i \Longrightarrow \Delta_i'$ for all $i \in I$.

We now use these weak derivations to define, in the standard manner of [13], weak action relations between derivations; these, together with the refusal relations $\stackrel{A}{\not\longrightarrow}$ for $A \subseteq \mathsf{Act}$ are the key ingredients in the definition of the failure-simulation preorder.

Definition 4.5 Let Δ and its variants $\Delta', \Delta^{\rm pre}, \Delta^{\rm post}$ be subdistributions in a pLTS $\langle S, \mathsf{Act}, \rightarrow \rangle$.

- For $a \in \mathsf{Act}$ write $\Delta \stackrel{a}{\Longrightarrow} \Delta'$ whenever $\Delta \Longrightarrow \Delta^{\rm pre} \xrightarrow{a} \Delta^{\rm post} \Longrightarrow \Delta'$, for some $\Delta^{\rm pre}$ and $\Delta^{\rm post}$. Extend this to Act_τ by allowing as a special case that $\stackrel{\tau}{\Longrightarrow}$ is simply $\Longrightarrow$, i.e. including identity (rather than requiring at least one $\xrightarrow{\tau}$).
- For $A \subseteq \mathsf{Act}$ and $s \in S$ write $s \stackrel{A}{\not\longrightarrow}$ if $s \stackrel{\alpha}{\not\longrightarrow}$ for every $\alpha \in A \cup \{\tau\}$; write $\Delta \stackrel{A}{\not\longrightarrow}$ if $s \stackrel{A}{\not\longrightarrow}$ for every $s \in \lceil \Delta \rceil$.
- More generally write $\Delta \Longrightarrow\stackrel{A}{\not\longrightarrow}$ if $\Delta \Longrightarrow \Delta^{\rm pre}$ for some $\Delta^{\rm pre}$ such that $\Delta^{\rm pre} \stackrel{A}{\not\longrightarrow}$.

Definition 4.6 [Failure simulation preorder] Define $\lhd_{FS}$ to be the largest relation in $S \times \mathscr{D}_{sub}(S)$ such that if $s \lhd_{FS} \Delta$ then

(i) whenever $\overline{s} \stackrel{\alpha}{\Longrightarrow} \Gamma'$, for $\alpha \in \mathsf{Act}_\tau$, then there is a $\Delta' \in \mathscr{D}_{sub}(S)$ with $\Delta \stackrel{\alpha}{\Longrightarrow} \Delta'$ and $\Gamma' \ \overline{\lhd_{FS}}\ \Delta'$,

(ii) and whenever $\overline{s} \Longrightarrow\stackrel{A}{\not\longrightarrow}$ then $\Delta \Longrightarrow\stackrel{A}{\not\longrightarrow}$.

Any relation $\mathscr{R} \subseteq S \times \mathscr{D}_{sub}(S)$ that satisfies the two clauses above is called a *failure simulation*. The failure simulation preorder $\sqsubseteq_{FS} \subseteq \mathscr{D}_{sub}(S) \times \mathscr{D}_{sub}(S)$ is defined by letting $\Delta \sqsubseteq_{FS} \Gamma$ whenever there is a $\Delta^\natural$ with $\Delta \Longrightarrow \Delta^\natural$ and $\Gamma \ \overline{\lhd_{FS}}\ \Delta^\natural$.

Note that the simulating process, Δ, occurs at the right of $\lhd_{FS}$, but at the left of $\sqsubseteq_{FS}$. The following lemma will bee needed in Section 6.

Lemma 4.7 If $\Gamma \,\overline{\lhd_{FS}}\, \Delta$ and $\Gamma \Longrightarrow \Gamma'$ then there is a matching transition $\Delta \Longrightarrow \Delta'$ such that $\Gamma' \,\overline{\lhd_{FS}}\, \Delta'$.

Proof: $\Gamma \,\overline{\lhd_{FS}}\, \Delta$ implies by Lemma 4.2 that $\qquad \Gamma = \sum_{i\in I} p_i \cdot \overline{s_i}, \qquad s_i \lhd_{FS} \Delta_i, \qquad \Delta = \sum_{i\in I} p_i \cdot \Delta_i.$

By Proposition 4.4(iii) there are $\Gamma'_i \in \mathscr{D}_{sub}(S)$ for $i\in I$ with $\overline{s_i} \Longrightarrow \Gamma'_i$ and $\Gamma' = \sum_{p_i\in I} p_i \cdot \Gamma'_i$. For each $i\in I$ we infer from $s_i \lhd_{FS} \Delta_i$ and $\overline{s_i} \Longrightarrow \Gamma'_i$ that there is a $\Delta'_i \in \mathscr{D}_{sub}(S)$ with $\Delta_i \Longrightarrow \Delta'_i$ and $\Gamma'_i \,\overline{\lhd_{FS}}\, \Delta'$. Let $\Delta' := \sum_{i\in I} p_i \cdot \Delta'_i$. Then Definition 4.1(2) and Proposition 4.4(ii) yield $\Gamma' \,\overline{\lhd_{FS}}\, \Delta'$ and $\Delta \Longrightarrow \Delta'$. □

The failure simulation preorder is preserved under parallel composition with a test and it is sound and complete for probabilistic must testing of finitary processes.

Theorem 4.8 [3] For finitary processes Δ and Γ,

(i) If $\Delta \sqsubseteq_{FS} \Gamma$ then for any Ω-test Θ, $\Theta \| \Delta \sqsubseteq_{FS} \Theta \| \Gamma$.

(ii) $\Delta \sqsubseteq_{FS} \Gamma$ if and only if $\Delta \sqsubseteq_{\text{pmust}} \Gamma$.

5 From derivations to resolutions

In this section we explain how resolutions, on which the definitions of the testing preorders in Definitions 3.4 and 3.5 are based, can be seen as certain kinds of derivations.

Definition 5.1 [Extreme derivatives] A state s in a pLTS is called *stable* if $s \stackrel{\tau}{\not\longrightarrow}$, and a subdistribution Δ is called *stable* if every state in its support is stable. We write $\Delta \Longrightarrow\!\!\!\succ \Delta'$ whenever $\Delta \Longrightarrow \Delta'$ and Δ' is stable, and call Δ' an *extreme* derivative of Δ.

Referring to Definition 4.3, we see this means that in the extreme derivation of Δ' from Δ at every stage a state must move on if it can, so that every stopping component can contain only states which *must* stop: for $s \in \lceil \Delta_k^{\rightarrow} + \Delta_k^{\times} \rceil$ we have $s \in \lceil \Delta_k^{\times} \rceil$ if *and now also* only if $s \stackrel{\tau}{\not\longrightarrow}$.

Lemma 5.2 [Existence and uniqueness of extreme derivatives]

(i) For every subdistribution Δ there exists some (stable) Δ' such that $\Delta \Longrightarrow\!\!\!\succ \Delta'$.

(ii) In a deterministic pLTS if $\Delta \Longrightarrow\!\!\!\succ \Delta'$ and $\Delta \Longrightarrow\!\!\!\succ \Delta''$ then $\Delta' = \Delta''$.

Proof: We construct a derivation as in Definition 4.3 of a stable Δ' by defining the components $\Delta_k, \Delta_k^{\times}$ and $\Delta_k^{\rightarrow}$ using induction on k. Let us assume that the subdistribution Δ_k has been defined; in the base case $k = 0$ this is simply Δ. The decomposition of this Δ_k into the components $\Delta_k^{\times}$ and $\Delta_k^{\rightarrow}$ is carried out by defining the former to be precisely those states which must stop, i.e. those s for which $s \stackrel{\tau}{\not\longrightarrow}$. Formally $\Delta_k^{\times}$ is determined by:

$$\Delta_k^{\times}(s) = \begin{cases} \Delta_k(s) & \text{if } s \stackrel{\tau}{\not\longrightarrow} \\ 0 & \text{otherwise} \end{cases}$$

Then $\Delta_k^{\rightarrow}$ is given by the *remainder* of Δ_k, namely those states which can perform a τ action:

$$\Delta_k^{\rightarrow}(s) = \begin{cases} \Delta_k(s) & \text{if } s \stackrel{\tau}{\longrightarrow} \\ 0 & \text{otherwise} \end{cases}$$

Note that these definitions divide the support of Δ_k into two disjoints sets, namely the support of $\Delta_k^{\times}$ and the support of $\Delta_k^{\rightarrow}$. Moreover by construction we know that $\Delta_k^{\rightarrow} \stackrel{\tau}{\longrightarrow} \Theta$ for some Θ; we let Δ_{k+1} be an arbitrary such Θ.

This completes our construction of an extreme derivative as in Definition 4.3 and so we have established (i).

For (ii) we observe that in a deterministic pLTS the above choice of Δ_{k+1} is unique, so that the whole derivative construction becomes unique. $\square$

*Sub*distributions are essential in the definition of extreme derivations. Consider a state t that has only one transition, a self τ-loop $t \xrightarrow{\tau} \bar{t}$. Then it diverges and it has a unique extreme derivative ε, the empty subdistribution. More generally, suppose a subdistribution Δ diverges, that is there is an infinite sequence of internal transitions $\Delta \xrightarrow{\tau} \Delta_1 \xrightarrow{\tau} \ldots \Delta_k \xrightarrow{\tau} \ldots$. Then one extreme derivative of Δ is ε, but it may have others.

In the extreme derivative $\Delta \Longrightarrow\!\!\succ \Delta'$, the subdistribution Δ' may be viewed as a final result of an execution starting in Δ and dynamically resolving nondeterministic choices as the execution proceeds. We can tabulate the outcome of this execution in the following manner:

Definition 5.3 [Outcomes] The outcome $\$\Phi \in [0,1]^{\Omega}$ of a stable subdistribution Φ is given by $\$\Phi(\omega) = \sum\{\Phi(s) \mid s \in \lceil\Phi\rceil,\ s \xrightarrow{\omega}\}$. For any distribution Φ we write $\mathcal{V}(\Phi)$ for the set of possible outcomes $\{\$\Phi' \mid \Phi \Longrightarrow\!\!\succ \Phi'\}$ via extreme derivatives.

Let $p_i \in [0,1]$ for $i \in I$ with $\sum_{i\in I} p_i \leq 1$, and let Δ_i, Φ_i, for $i \in I$, be subdistributions. We use $\sum_{i\in I} p_i \cdot \mathcal{V}(\Delta_i)$ as shorthand for $\{\sum_{i\in I} p_i \cdot v_i \mid v_i \in \mathcal{V}(\Delta_i)\}$. By construction, $\$\sum_{i\in I} p_i \cdot \Phi_i = \sum_{i\in I} p_i \cdot \$\Phi_i$. Using this, we establish the linearity of $\mathcal{V}$:

Lemma 5.4 Let $p_i \in [0,1]$ for $i \in I$ with $\sum_{i\in I} p_i \leq 1$. Then $\mathcal{V}(\sum_{i\in I} p_i \cdot \Delta_i) = \sum_{i\in I} p_i \cdot \mathcal{V}(\Delta_i)$.

Proof: Suppose $v \in \mathcal{V}(\sum_{i\in I} p_i \cdot \Delta_i)$. Then $v = \$\Phi$ for some stable Φ with $\sum_{i\in I} p_i \cdot \Delta_i \Longrightarrow \Phi$. By Proposition 4.4(iii) Φ can be written as $\sum_{i\in I} p_i \cdot \Phi_i$ for subdistributions Φ_i such that $\Delta_i \Longrightarrow \Phi_i$ for all $i \in I$; moreover, the Φ_i must be stable. Hence $v_i := \$\Phi_i \in \mathcal{V}(\Delta_i)$ and thus $v = \sum_{i\in I} p_i \cdot v_i \in \sum_{i\in I} p_i \cdot \mathcal{V}(\Delta_i)$.

Conversely, suppose $v \in \sum_{i\in I} p_i \cdot \mathcal{V}(\Delta_i)$, i.e., $v = \sum_{i\in I} p_i \cdot v_i$ with $v_i \in \mathcal{V}(\Delta_i)$. Then for all $i \in I$ there are stable subdistributions Φ_i with $v_i := \$\Phi_i$ and $\Delta_i \Longrightarrow \Phi_i$. So $\sum_{i\in I} p_i \cdot \Delta_i \Longrightarrow \sum_{i\in I} p_i \cdot \Phi_i$ by Proposition 4.4(ii). Moreover $\sum_{i\in I} p_i \cdot \Phi_i$ is stable and $v = \sum_{i\in I} p_i \cdot v_i = \$\sum_{i\in I} p_i \cdot \Phi_i \in \mathcal{V}(\sum_{i\in I} p_i \cdot \Delta_i)$. $\square$

The following two examples illustrate that this manner of calculating outcomes often gives the same result as when resolutions are used.

Example 5.5 (Revisiting Example 3.2.) The pLTS in Figure 4(c) is deterministic and therefore from part (ii) of Lemma 5.2 it follows that $\overline{t \| q_1}$ has a unique extreme derivative Λ. Moreover Λ can be calculated to be $\sum_{k\geq 1} \frac{1}{2^k} \cdot \overline{s_3}$, which simplifies to the distribution $\overline{s_3}$. Therefore, since $\$\overline{s_3} = \overrightarrow{\omega}$, it follows that $\mathcal{V}(\overline{t \| q_1}) = \{\overrightarrow{\omega}\}$. This is exactly the same result as obtained in Example 3.2, using resolutions. $\square$

Example 5.6 (Revisiting Example 3.3.) The application of the test $\bar{t}$ to processes $\overline{q_2}$ is outlined in Figure 5(c). Consider any extreme derivative Δ' from $s_0 = \bar{t} \| \overline{q_2}$. Using the notation of Definition 4.3, it is clear that $\Delta_0^{\times}$ and $\Delta_0^{\rightarrow}$ must be ε and $\overline{s_0}$ respectively. Similarly, $\Delta_1^{\times}$ and $\Delta_1^{\rightarrow}$ must be ε and $\overline{s_1}$ respectively. But s_1 is a nondeterministic state, having two possible transitions:

(i) $s_1 \xrightarrow{\tau} \Lambda_0$ where Λ_0 has support $\{s_0, s_2\}$ and assigns each of them the weight $\frac{1}{2}$

(ii) $s_1 \xrightarrow{\tau} \Lambda_1$ where Λ_1 has the support $\{s_3, s_4\}$, again dividing the mass equally among them.

So there are many possibilities for Δ_2; from Definition 4.3 one sees that in fact Δ_2 can be of the form

$$p \cdot \Lambda_0 + (1-p) \cdot \Lambda_1 \tag{7}$$

for any choice of $p \in [0,1]$.

Let us consider one possibility, an extreme one where p is chosen to be 0; only the transition (ii) above is used. Here $\Delta_2^{\rightarrow}$ is the subdistribution $\frac{1}{2}\overline{s_4}$, and $\Delta_k^{\rightarrow} = \varepsilon$ whenever $k > 2$. A simple calculation shows that in this case the extreme derivative generated is $\Lambda_1^e = \frac{1}{2}\overline{s_3} + \frac{1}{2}\overline{s_6}$ which implies that $\frac{1}{2}\overrightarrow{\omega} \in \mathcal{V}(\bar{t}\|\overline{q_2})$.

Another possibility for Δ_2 is Λ_0, corresponding to $p = 1$ in (7) above. Continuing this derivation leads to Δ_3 being $\frac{1}{2}\cdot\overline{s_1} + \frac{1}{2}\cdot\overline{s_5}$; thus $\Delta_3^{\times} = \frac{1}{2}\cdot\overline{s_5}$ and $\Delta_3^{\rightarrow} = \frac{1}{2}\cdot\overline{s_1}$. Now in the generation of Δ_4 from $\Delta_3^{\rightarrow}$ again we resolve a transition from the nondeterministic state s_1, by choosing some arbitrary $p \in [0,1]$ in (7). Suppose we choose $p = 1$ every time, completely ignoring transition (ii) above. Then the extreme derivative generated is

$$\Lambda_0^e = \sum_{k\geq 1} \frac{1}{2^k}\cdot\overline{s_5}$$

which simplifies to the distribution $\overline{s_5}$. This in turn means that $\overrightarrow{\omega} \in \mathcal{V}(\bar{t}\|\overline{q_2})$.

We have seen two possible derivations of extreme derivatives from $\overline{s_0}$. But there are many others. In general whenever $\Delta_k^{\rightarrow}$ is of the form $q\cdot\overline{s_1}$ we have to resolve the nondeterminism by choosing a $p \in [0,1]$ in (7) above; moreover each such choice is independent. It turns out that every extreme derivative Δ' of $\overline{s_0}$ is of the form $q\cdot\Lambda_0^e + (1-q)\cdot\Lambda_1^e$ for some choice of $q \in [0,1]$, which implies that $\mathcal{V}(\bar{t}\|\overline{q_2})$ is the convex closure of the set $\{\frac{1}{2}\overrightarrow{\omega}, \overrightarrow{\omega}\}$.

Again this is similar to the results obtained using resolutions, in Example 3.3. □

Unfortunately there is not an exact agreement between using resolutions and extreme derivations, as the next example shows.

Example 5.7 Let $\overline{p}$ be a process that first does an a-action, to the point distribution $\overline{q}$, and then diverges, via the τ-loop $q \xrightarrow{\tau} \overline{q}$. Let $\bar{t}$ be the test used in Examples 3.2 and 3.3. It is easy to see that the distribution $\overline{p\|t}$ has a unique resolution, with expected outcome $\overrightarrow{\omega}$; thus $\mathcal{A}(\bar{t},\overline{p}) = \{\overrightarrow{\omega}\}$.

It turns out that $\bar{t}\|\overline{p}$ also has a unique extreme derivative; unfortunately this turns out to be ε. Since $\$\varepsilon = 0$ this means that $\mathcal{V}(\bar{t}\|\overline{p}) = \overrightarrow{0}$; so in this case, which is actually nonprobabilistic, there is a difference between the use of resolutions and extreme derivations. □

To rectify this anomaly, we restrict our attention to a subset of pLTSs.

Definition 5.8 [ω-respecting] A pLTS $\langle S,\Omega,\rightarrow\rangle$ is said to be ω*-respecting* when it satisfies the uniqueness requirement (A) from Page 5, and $s \xrightarrow{\omega}$, for any $\omega \in \Omega$, implies $s \not\xrightarrow{\tau}$.

It is straightforward to modify the pLTS of applications of tests to processes into one that it is ω-respecting, namely by removing all transitions $s \xrightarrow{\tau} \Delta$ for states s with $s \xrightarrow{\omega}$; we call this *pruning*. We denote the result of pruning the pLTS $\langle S,\Omega,\rightarrow\rangle$ by $\langle S,\Omega,[\rightarrow]\rangle$, and the distribution Φ in this pruned pLTS by $[\Phi]$.

Example 5.9 (Revisiting Example 5.7) Let $\overline{p},\overline{q}$ and $\bar{t}$ be as in Example 5.7. As we have already seen, $\bar{t}\|\overline{p}$ has the unique derivative ε. But by pruning it we obtain a different extreme derivative. If we denote the state reachable from $\bar{t}$ with the outgoing ω-transition, in Figure 5(c), as ω also, then $[\bar{t}\|\overline{p}]$ has the unique extreme derivative $[\omega\|q]$. Since $\$[\omega\|q] = \overrightarrow{\omega}$, we obtain $\mathcal{V}([\bar{t}\|\overline{p}]) = \{\overrightarrow{\omega}\}$; this is exactly the result obtained using resolutions. □

Note that pruning has no effect on Examples 5.5 and 5.6, as the pLTSs concerned are already ω-respecting. It also has no effect on the closure of the failure simulation preorder under parallel composition:

Lemma 5.10 [4] For finitary processes Δ and Γ, if $\Delta \sqsubseteq_{FS} \Gamma$ then for any Ω-test Θ, $[\Theta\|\Delta] \sqsubseteq_{FS} [\Theta\|\Gamma]$.

In the remainder of this section we show that, at least in ω-respecting pLTSs, using resolutions to calculate outcomes, as used in the definition of testing (Definitions 3.4 and 3.5), leads to the same results as using extreme derivations. In the former a set of deterministic structures are associated with a distribution, while in the latter nondeterministic choices are resolved dynamically as the derivation proceeds. We start by showing that resolution-based testing is insensitive to pruning. Let $\mathscr{A}^{\mathrm{p}}(\Phi)$ denote the set of vectors

$$\{\,\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle}) \mid \langle R,\Lambda,\rightarrow_R\rangle \text{ is a resolution of } [\Phi]\,\}\ .$$

Proposition 5.11 For any distribution Φ in a pLTS $\langle S,\Omega,\rightarrow\rangle$ we have that $\mathscr{A}^{\mathrm{p}}(\Phi)=\mathscr{A}(\Phi)$.

Proof: "$\supseteq$": Let $\langle R,\Lambda,\rightarrow_R\rangle$ be a resolution of Φ. Then, following Definition 3.1, $\langle R,[\Lambda],[\rightarrow_R]\rangle$ is a resolution of $[\Phi]$ and, by (3), $\mathrm{Exp}_{[\Lambda]}(\mathbb{V}_{\langle R,\Omega,[\rightarrow_R]\rangle})=\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle})$.

"$\subseteq$": Let $\langle R,\Lambda,\rightarrow_R\rangle$ be a resolution of $[\Phi]$ with resolving function f. We construct a resolution $\langle R',\Lambda,\rightarrow'_R\rangle$ of Φ as a random extension of $\langle R,\Lambda,\rightarrow_R\rangle$. For every pair $(s,\alpha)\in S\times\Omega_\tau$ with $s\xrightarrow{\alpha}$ pick a distribution $\Psi^{(s,\alpha)}\in\mathscr{D}(S)$ such that $s\xrightarrow{\alpha}\Psi^{(s,\alpha)}$. Now define $R':=R\,\dot\cup\,(S\times\mathbb{N})$, where $\dot\cup$ denotes the disjoint union operation, and obtain $\rightarrow'_R$ from $\rightarrow_R$ by adding (A) a transition $(s,k)\xrightarrow{\alpha}{}'_R\,\Psi^{(s,\alpha)}_{k+1}$ for each $k\in\mathbb{N}$ and each $s\in S$ with $s\xrightarrow{\alpha}$, and (B) a transition $r\xrightarrow{\tau}{}'_R\,\Psi_0^{(f(r),\tau)}$ for each $r\in R$ with $f(r)\xrightarrow{\tau}$ as well as $f(r)\xrightarrow{\omega}$ for some $\omega\in\Omega$. Here $\Psi^{(s,\alpha)}_{k+1}\in\mathscr{D}(S\times\{k{+}1\})$ is given by $\Psi^{(s,\alpha)}_{k+1}(t,k{+}1)=\Psi^{(s,\alpha)}(t)$ for all $t\in S$. The resolving function f is extended by $f(s,k):=s$. Using Definition 3.1 it follows that $\langle R',\Lambda,\rightarrow'_R\rangle$ is a resolution of Φ and, again by (3), $\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R',\Omega,\rightarrow'_R\rangle})=\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle})$. □

The rest of this section is devoted to showing that $\mathscr{V}([\Phi])=\mathscr{A}^{\mathrm{p}}(\Phi)$ for any composition $\Phi=\Theta\|\Delta$ of a test Θ and process Δ; this amounts to showing

$$\{\$\Phi'\mid\Phi\Longrightarrow\!\!\succ\Phi'\}=\{\,\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow\rangle})\mid\langle R,\Lambda,\rightarrow\rangle\text{ is a resolution of }\Phi\,\}$$

for any distribution Φ in an ω-respecting pLTS $\langle S,\Omega,\rightarrow\rangle$.

Let us see how an extreme derivation can be viewed as a method for dynamically generating a resolution.

Proposition 5.12 [Resolutions from extreme derivatives] Let $\Phi\Longrightarrow\!\!\succ\Phi'$ in a pLTS $\langle S,\Omega,\rightarrow\rangle$. Then there is a resolution $\langle R,\Lambda,\rightarrow_R\rangle$ of Φ, with resolving function f, such that $\Lambda\Longrightarrow\!\!\succ_R\Lambda'$ for some Λ' for which $\Phi'=\mathrm{Img}_f(\Lambda')$.

Proof: Consider an extreme derivation of $\Phi\Longrightarrow\!\!\succ\Phi'$ as given in Definition 4.3 where all $\Phi^\times_k$ must be stable:

$$\Phi=\Phi_0,\qquad\Phi_k=\Phi^\times_k+\Phi^\rightarrow_k,\qquad\Phi^\rightarrow_k\xrightarrow{\tau}\Phi_{k+1},\qquad\Phi'=\textstyle\sum_{k=0}^\infty\Phi^\times_k.$$

By Lemma 4.2, $\Phi^\rightarrow_k\xrightarrow{\tau}\Phi_{k+1}$ implies that there are states $s_{ik}\in S$ and distributions $\Phi_{i(k+1)}\in\mathscr{D}(S)$, such that

$$\Phi^\rightarrow_k=\textstyle\sum_{i\in I_k}p_{ik}\cdot\overline{s_{ik}},\qquad s_{ik}\xrightarrow{\tau}\Phi_{i(k+1)}\ \text{ for each } i\in I_k\qquad\text{and}\qquad\Phi_{k+1}=\textstyle\sum_{i\in I_k}p_{ik}\cdot\Phi_{i(k+1)}\ .$$

Let $\Phi^\times_{ik}(s):=\begin{cases}\Phi_{ik}(s)&\text{if } s\not\xrightarrow{\tau}\\0&\text{if } s\xrightarrow{\tau}\end{cases}$. Since $\Phi^\times_k(s)=\begin{cases}\Phi_k(s)&\text{if } s\not\xrightarrow{\tau}\\0&\text{if } s\xrightarrow{\tau}\end{cases}$ it follows that $\Phi^\times_{k+1}=\sum_{i\in I_k}p_{ik}\cdot\Phi^\times_{i(k+1)}$.

We will now define the resolution $\langle R,\Lambda,\rightarrow_R\rangle$ and the resolving function f. The set of states R is $(S\times\mathbb{N})\cup\bigcup_{k\in\mathbb{N}}(I_k\times\{k\})$. The resolving function $f:R\rightarrow S$ maps $(s,k)\in S\times\mathbb{N}$ to s and $(i,k)\in I_k\times\{k\}$ to $s_{ik}\in S$. The second component k of a state counts how many transitions have fired already: each transition in $\rightarrow_R$ goes from a state (i,k) or (s,k) to a distribution over $(S\cup I_{k+1})\times\{k{+}1\}$.

Define the subdistributions $\Lambda^\times_k\in\mathscr{D}_{sub}(S\times\{k\})$ and $\Lambda^\rightarrow_k\in\mathscr{D}_{sub}(I_k\times\{k\})$ by $\Lambda^\times_k(s,k)=\Phi^\times_k(s)$ and $\Lambda^\rightarrow_k(i,k)=p_{ik}$. Let $\Lambda_k:=\Lambda^\times_k+\Lambda^\rightarrow_k$ and $\Lambda:=\Lambda_0$. Furthermore, for all $k>0$ and $i\in I_{k-1}$, define

$\Lambda_{ik} \in \mathscr{D}_{sub}((S \cup I_k) \times \{k\})$ by

$$\Lambda_{ik}(s,k) = \Phi^{\times}_{ik}(s) \qquad \text{and} \qquad \Lambda_{ik}(j,k) = p_{jk} \cdot \frac{\Phi_{ik}(s_{jk})}{\Phi_k(s_{jk})}$$

for $j \in I_k$. We introduce the transitions $(i,k) \xrightarrow{\tau}_R \Lambda_{i(k+1)}$ for $k \geq 0$ and $i \in I_k$. Moreover, for each state $s \in S$ and label $\alpha \in \mathsf{Act}_\tau$ such that $s \xrightarrow{\alpha}$, pick a transition $s \xrightarrow{\alpha} \Psi$, and add the transition $(s,k) \xrightarrow{\alpha}_R \Psi_{k+1}$ to $\rightarrow_R$, for all $k \in \mathbb{N}$. Here Ψ_{k+1} is the distribution with $\Psi_{k+1}(t,k+1) = \Psi(t)$ for all $t \in S$. Likewise, for each $k \in \mathbb{N}$, $i \in I_k$ and $\omega \in \Omega$ such that $s_{ik} \xrightarrow{\omega}$, pick a transition $s_{ik} \xrightarrow{\omega} \Psi$, and add the transition $(i,k) \xrightarrow{\omega}_R \Psi_{k+1}$ to $\rightarrow_R$. This ends the definition of the resolution $\langle R, \Lambda, \rightarrow_R \rangle$ and the resolving function f. By construction, $\langle R, \Omega, \rightarrow_R \rangle$ is a deterministic pLTS. We now check that f satisfies the requirements for a resolving function of Definition 3.1.

(i) $$\mathrm{Img}_f(\Lambda_k)(s) = \Lambda_k(s,k) + \sum_{s_{ik}=s} \Lambda_k(i,k) = \Lambda^{\times}_k(s,k) + \sum_{s_{ik}=s} p_{ik} = \Phi^{\times}_k(s) + \Phi^{\rightarrow}_k(s) = \Phi_k(s)$$

for all $s \in S$, so $\mathrm{Img}_f(\Lambda_k) = \Phi_k$, and in particular $\mathrm{Img}_f(\Lambda) = \Phi$.

(ii) Let $r \xrightarrow{\alpha}_R \Gamma$ for $\alpha \in \Omega_\tau$. In case $r = (s,k)$ it must be that $\Gamma = \Psi_{k+1}$ and $f(r) = s \xrightarrow{\alpha} \Psi = \mathrm{Img}_f(\Psi_{k+1})$. Likewise, in case $r = (i,k)$ and $\alpha \in \Omega$ it must be that $\Gamma = \Psi_{k+1}$ and $f(r) = s_{ik} \xrightarrow{\alpha} \Psi = \mathrm{Img}_f(\Psi_{k+1})$. The remaining case is $r = (i,k)$, $\alpha = \tau$ and $\Gamma = \Lambda_{i(k+1)}$. Then $f(r) = s_{ik} \xrightarrow{\tau} \Phi_{i(k+1)}$, so it suffices to show that $\mathrm{Img}_f(\Lambda_{ik}) = \Phi_{ik}$ for all $k \in \mathbb{N}$ and $i \in I_k$. For any $s \in S$ we have

$$\mathrm{Img}_f(\Lambda_{ik})(s) = \Lambda_{ik}(s,k) + \sum_{s_{jk}=s} \Lambda_{ik}(j,k) = \Phi^{\times}_{ik}(s) + \sum_{s_{jk}=s} p_{jk} \cdot \frac{\Phi_{ik}(s_{jk})}{\Phi_k(s_{jk})} = \Phi^{\times}_{ik}(s) + \frac{\Phi_{ik}(s)}{\Phi_k(s)} \cdot \sum_{s_{jk}=s} p_{jk}\,.$$

In case $s \not\xrightarrow{\tau}$ we have $s_{jk} = s$ for no $j \in I_k$, so $\mathrm{Img}_f(\Lambda_{ik})(s) = \Phi^{\times}_{ik}(s) = \Phi_{ik}(s)$.
In case $s \xrightarrow{\tau}$ we have $\Phi^{\times}_{ik}(s) = 0$ and $\sum_{s_{jk}=s} p_{jk} = \Phi^{\rightarrow}_k(s) = \Phi_k(s)$, so again $\mathrm{Img}_f(\Lambda_{ik})(s) = \Phi_{ik}(s)$.

(iii) Let $f(r) \xrightarrow{\alpha}$ for $\alpha \in \Omega_\tau$. By construction there is a Ψ_{k+1} such that $r \xrightarrow{\alpha}_R \Psi_{k+1}$.

Hence $\langle R, \Lambda, \rightarrow_R \rangle$ is a resolution of Φ. We have:

$$\sum_{i \in I_k} p_{ik} \cdot \Lambda_{i(k+1)}(s,k{+}1) = \sum_{i \in I_k} p_{ik} \cdot \Phi^{\times}_{i(k+1)}(s) = \Phi^{\times}_{k+1}(s) = \Lambda^{\times}_{k+1}(s,k{+}1) = \Lambda_{k+1}(s,k{+}1)$$

$$\sum_{i \in I_k} p_{ik} \cdot \Lambda_{i(k+1)}(j,k{+}1) = \sum_{i \in I_k} p_{ik} \cdot p_{j(k+1)} \cdot \frac{\Phi_{i(k+1)}(s_{j(k+1)})}{\Phi_{k+1}(s_{j(k+1)})} = p_{j(k+1)} = \Lambda^{\rightarrow}_{k+1}(j,k{+}1) = \Lambda_{k+1}(j,k{+}1).$$

Hence $\Lambda_{k+1} = \sum_{i \in I_k} p_{ik} \cdot \Lambda_{i(k+1)}$. Since also $\Lambda^{\rightarrow}_k = \sum_{i \in I_k} p_{ik} \cdot \overline{(i,k)}$ and $(i,k) \xrightarrow{\tau}_R \Lambda_{i(k+1)}$, Lemma 4.2 yields $\Lambda^{\rightarrow}_k \xrightarrow{\tau}_R \Lambda_{k+1}$. Let $\Lambda' = \sum_{k=0}^{\infty} \Lambda^{\times}_k$. Then, by Definition 4.3, $\Lambda \Longrightarrow\!\!\succ_R \Lambda'$.

By construction $\mathrm{Img}_f(\Lambda^{\times}_k) = \Phi^{\times}_k$ for all $k \in \mathbb{N}$. Hence $\mathrm{Img}_f(\Lambda') = \sum_{k=0}^{\infty} \mathrm{Img}_f(\Lambda^{\times}_k) = \sum_{k=0}^{\infty} \Phi^{\times}_k = \Phi'$. □

The converse is somewhat simpler.

Proposition 5.13 [Extreme derivatives from resolutions] Let $\langle R, \Lambda, \rightarrow_R \rangle$ be a resolution of a subdistribution Φ in a pLTS $\langle S, \Omega, \rightarrow \rangle$ with resolving function f. Then $\Lambda \Longrightarrow\!\!\succ_R \Lambda'$ implies $\Phi \Longrightarrow\!\!\succ \mathrm{Img}_f(\Lambda')$.

Proof: The definition of Img_f implies that $\mathrm{Img}_f(\sum_i p_i \cdot \Psi_i) = \sum_i p_i \cdot \mathrm{Img}_f(\Psi_i)$. Furthermore $\Psi \xrightarrow{\tau} \Psi'$ implies $\mathrm{Img}_f(\Psi) \xrightarrow{\tau} \mathrm{Img}_f(\Psi')$. Namely, by Lemma 4.2, $\Psi \xrightarrow{\tau} \Psi'$ implies

$$\Psi = \textstyle\sum_{i \in I} p_i \cdot \overline{s_i}, \qquad s_i \xrightarrow{\tau} \Psi_i \ \text{ for each } i \in I \qquad \text{and} \qquad \Psi' = \textstyle\sum_{i \in I} p_i \cdot \Psi_i$$

which, using Definition 3.1, entails

$$\mathrm{Img}_f(\Psi) = \sum_{i\in I} p_i \cdot \overline{f(s_i)},\;\; f(s_i) \xrightarrow{\tau} \mathrm{Img}_f(\Psi_i) \;\text{ for each } i \in I \;\text{ and }\; \mathrm{Img}_f(\Psi') = \sum_{i\in I} p_i \cdot \mathrm{Img}_f(\Psi_i).$$

Hence $\mathrm{Img}_f(\Psi) \xrightarrow{\tau} \mathrm{Img}_f(\Psi')$.

Now consider any derivation of $\Lambda \Longrightarrow\!\!\succ_R \Lambda'$ along the lines of Definition 4.3. By systematically applying the function f to the component subdistributions in this derivation we get a derivation $\mathrm{Img}_f(\Lambda) \Longrightarrow \mathrm{Img}_f(\Lambda')$, that is $\Phi \Longrightarrow \mathrm{Img}_f(\Lambda')$. To show that $\mathrm{Img}_f(\Lambda')$ is actually an extreme derivative it suffices to show that s is stable for every $s \in \lceil \mathrm{Img}_f(\Lambda') \rceil$. But if $s \in \lceil \mathrm{Img}_f(\Lambda') \rceil$ then by definition there is some $t \in \lceil \Lambda' \rceil$ such that $s = f(t)$. Since $\Lambda \Longrightarrow\!\!\succ_R \Lambda'$ the state t must be stable. The stability of s now follows from requirement (iii) of Definition 3.1. □

Our next step is to relate the outcomes extracted from extreme derivatives to those extracted from the corresponding resolutions. This requires some analysis of the evaluation function $\mathbb{V}$ applied to ω-respecting deterministic pLTSs. We show that the function $\mathscr{F}$ defined in (3) on Page 6 and its least fixed point $\mathbb{V}$ are continuous with respect to the standard Euclidean metric.

Definition 5.14 [Continuous functions] An ω-chain in a complete lattice L is a sequence of elements $\{c_n \mid n \geq 0\}$ satisfying $c_i \leq c_{i+1}$. Since the lattice is complete, ω-chains have least upper bounds; we denote them by $\bigsqcup_{n\geq 0} c_n$. A function $f : L \to L$ is said to be (ω)-continuous [19] if it preserves the least upper bounds of ω-chains:

$$f(\bigsqcup_{n\geq 0} c_n) = \bigsqcup_{n\geq 0} f(c_n)\;.$$

Lemma 5.15 [Exchange of suprema] Let function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be such that it is

(i) monotonic in both of its arguments separately, so that $i \leq i'$ implies $g(i, j) \leq g(i', j)$ for all j, and $j \leq j'$ implies $g(i, j) \leq g(i, j')$ for all i, and

(ii) bounded above, so that there is a $c \in \mathbb{R}_{\geq 0}$ with $g(i, j) \leq c$ for all i, j.

Then

$$\lim_{i\to\infty} \lim_{j\to\infty} g(i, j) \;\;=\;\; \lim_{j\to\infty} \lim_{i\to\infty} g(i, j).$$

Proof: Conditions (i) and (ii) guarantee the existence of all the limits. Moreover, for a non-decreasing sequence its limit and supremum agree, and both sides equal the supremum of all $g(i, j)$ for $i, j \in \mathbb{N}$. In fact, $(\mathbb{R}, \leq)$ is a complete partial order (CPO), and it is a basic result of CPOs [19] that

$$\bigsqcup_{i\in\mathbb{N}} (\bigsqcup_{j\in\mathbb{N}} g(i, j)) \;\;=\;\; \bigsqcup_{j\in\mathbb{N}} (\bigsqcup_{i\in\mathbb{N}} g(i, j)).$$ □

The following technical proposition states that some real functions satisfy the property of *bounded continuity*, which allows the exchange of limit and sum operations. It plays a crucial role in proving the continuity of $\mathscr{F}$.

Proposition 5.16 [Bounded continuity] Given a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions:

C1. f is monotonic in the second parameter, i.e. $j_1 \leq j_2$ implies $f(i, j_1) \leq f(i, j_2)$ for all $i, j_1, j_2 \in \mathbb{N}$;

C2. for any $i \in \mathbb{N}$, the limit $\lim_{j\to\infty} f(i, j)$ exists;

C3. the partial sums $S_n = \sum_{i=0}^{n} \lim_{j\to\infty} f(i, j)$ are bounded, i.e. there exists some $c \in \mathbb{R}_{\geq 0}$ such that $S_n \leq c$ for all $n \geq 0$;

then it holds that

$$\sum_{i=0}^{\infty} \lim_{j\to\infty} f(i,j) \;=\; \lim_{j\to\infty} \sum_{i=0}^{\infty} f(i,j).$$

Proof: Let $g : \mathbb{N}\times\mathbb{N} \to \mathbb{R}_{\geq 0}$ be the function defined by $g(n,j) = \sum_{i=0}^{n} f(i,j)$. It is easy to see that g is monotonic in both arguments. By **C1** and **C2**, we have that $f(i,j) \leq \lim_{j\to\infty} f(i,j)$ for any $i,j \in \mathbb{N}$. So for any $j,n \in \mathbb{N}$ we have that

$$g(n,j) = \sum_{i=0}^{n} f(i,j) \;\leq\; \sum_{i=0}^{n} \lim_{j\to\infty} f(i,j) \;\leq\; c$$

according to **C3**. In other words, g is bounded above. Therefore we can apply Lemma 5.15 and obtain

$$\lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=0}^{n} f(i,j) \;=\; \lim_{j\to\infty} \lim_{n\to\infty} \sum_{i=0}^{n} f(i,j). \tag{8}$$

For any $j \in \mathbb{N}$, the sequence $\{g(n,j)\}_{n\geq 0}$ is nondecreasing and bounded, so its limit $\sum_{i=0}^{\infty} f(i,j)$ exists. That is,

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(i,j) \;=\; \sum_{i=0}^{\infty} f(i,j). \tag{9}$$

In view of **C2**, we have that, for any given $n \in \mathbb{N}$, the limit $\lim_{j\to\infty} \sum_{i=0}^{n} f(i,j)$ exists and

$$\sum_{i=0}^{n} \lim_{j\to\infty} f(i,j) \;=\; \lim_{j\to\infty} \sum_{i=0}^{n} f(i,j). \tag{10}$$

By **C3** the sequence $\{S_n\}_{n\geq 0}$ is bounded. Since it is also nondecreasing, it converges to $\displaystyle\sum_{i=0}^{\infty} \lim_{j\to\infty} f(i,j)$. That is,

$$\lim_{n\to\infty} \sum_{i=0}^{n} \lim_{j\to\infty} f(i,j) \;=\; \sum_{i=0}^{\infty} \lim_{j\to\infty} f(i,j). \tag{11}$$

Hence the left-hand side of the desired equality exists. By combining (8)-(11) we obtain the result that $\displaystyle\sum_{i=0}^{\infty} \lim_{j\to\infty} f(i,j) \;=\; \lim_{j\to\infty} \sum_{i=0}^{\infty} f(i,j)$. □

Lemma 5.17 Let R be a set and $h : R \to [0,1]^{\Omega}$. Furthermore, let $\Delta_0 \leq \Delta_1 \leq \cdots$ be an ω-chain of subdistributions over R — here $\Delta \leq \Delta'$ iff $\Delta(r) \leq \Delta'(r)$ for all $r \in R$. Then $\mathrm{Exp}_{\bigsqcup_{n\geq 0}\Delta_n} h = \bigsqcup_{n\geq 0} \mathrm{Exp}_{\Delta_n} h$.

Proof:
$$\begin{aligned}
\big(\mathrm{Exp}_{\bigsqcup_{n\geq 0}\Delta_n} h\big)(\omega) &= \big(\textstyle\sum_{r\in R} (\bigsqcup_{n\geq 0}\Delta_n)(r)\cdot h(r)\big)(\omega)\\
&= \big(\textstyle\sum_{r\in R} (\bigsqcup_{n\geq 0}\Delta_n(r))\cdot h(r)\big)(\omega)\\
&= \big(\textstyle\sum_{r\in R} \bigsqcup_{n\geq 0}(\Delta_n(r)\cdot h(r))\big)(\omega)\\
&= \textstyle\sum_{r\in R} \bigsqcup_{n\geq 0}(\Delta_n(r)\cdot h(r)(\omega))\\
&= \textstyle\sum_{r\in R} \lim_{n\to\infty}(\Delta_n(r)\cdot h(r)(\omega))\\
&= \textstyle\lim_{n\to\infty} \sum_{r\in R}(\Delta_n(r)\cdot h(r)(\omega)) \quad \text{by Proposition 5.16}\\
&= \textstyle\bigsqcup_{n\geq 0} \sum_{r\in R}(\Delta_n(r)\cdot h(r)(\omega))\\
&= \big(\textstyle\bigsqcup_{n\geq 0} \sum_{r\in R}(\Delta_n(r)\cdot h(r))\big)(\omega)\\
&= \big(\textstyle\bigsqcup_{n\geq 0} \mathrm{Exp}_{\Delta_n} h\big)(\omega).
\end{aligned}$$

In the above reasoning, Proposition 5.16 can be applied because we can define $f : R\times\mathbb{N} \to \mathbb{R}_{\geq 0}$ by letting $f(r,n) = \Delta_n(r)\cdot h(r)(\omega)$ and checking that f satisfies the three conditions in Proposition 5.16. If R is finite, we can extend it to a countable set $R' \supseteq R$ and require $f(r',n) = 0$ for all $r' \in R'\backslash R$ and $n \in \mathbb{N}$.

1. f satisfies condition **C1**. For any $r \in R$ and $j_1, j_2 \in \mathbb{N}$, if $j_1 \leq j_2$ then $\Delta_{j_1} \leq \Delta_{j_2}$. It follows that
$$f(r, j_1) \;=\; \Delta_{j_1}(r) \cdot h(r)(\omega) \;\leq\; \Delta_{j_2}(r) \cdot h(r)(\omega) \;=\; f(r, j_2).$$
2. f satisfies condition **C2**. For any $r \in R$, the sequence $\{\Delta_n(r) \cdot h(r)(\omega)\}_{n \geq 0}$ is nondecreasing and bounded by $h(r)(\omega)$. It follows that the limit $\lim_{n\to\infty} f(r,n)$ exists.
3. f satisfies condition **C3**. For any finite $R'' \subseteq R$, the partial sum $\sum_{r\in R''} \lim_{n\to\infty} f(r,n)$ is bounded because
$$\begin{array}{lclcl} \sum_{r\in R''} \lim_{n\to\infty} f(r,n) & = & \lim_{n\to\infty} \sum_{r\in R''} f(r,n) & = & \lim_{n\to\infty} \sum_{r\in R''} \Delta_n(r) \cdot h(r)(\omega) \\ \leq \;\; \lim_{n\to\infty} \sum_{r\in R''} \Delta_n(r) & \leq & \lim_{n\to\infty} \sum_{r\in R} \Delta_n(r) & \leq & \lim_{n\to\infty} 1 = 1. \end{array}$$
□

Lemma 5.18 Consider a deterministic pLTS $\langle R, \Omega, \rightarrow\rangle$. The function $\mathscr{F}$ defined in (3) is continuous.

Proof: Let $f_0 \leq f_1 \leq ...$ be an increasing chain in $R \rightarrow [0,1]^\Omega$. We need to show that
$$\mathscr{F}(\bigsqcup_{n\geq 0} f_n) \;=\; \bigsqcup_{n\geq 0} \mathscr{F}(f_n) \tag{12}$$
For any $r \in R$, we are in one of the following three cases:

1. $r \xrightarrow{\omega}$ for some $\omega \in \Omega$. We have
$$\begin{array}{lcll} \mathscr{F}(\bigsqcup_{n\geq 0} f_n)(r)(\omega) & = & 1 & \text{by (3)} \\ & = & \bigsqcup_{n\geq 0} 1 & \\ & = & \bigsqcup_{n\geq 0} \mathscr{F}(f_n)(r)(\omega) & \\ & = & (\bigsqcup_{n\geq 0} \mathscr{F}(f_n))(r)(\omega) & \end{array}$$
and
$$\mathscr{F}(\bigsqcup_{n\geq 0} f_n)(r)(\omega') = 0 = (\bigsqcup_{n\geq 0} \mathscr{F}(f_n))(r)(\omega')$$
for all $\omega' \neq \omega$.
2. $r \not\rightarrow$. Similar to the last case. We have
$$\mathscr{F}(\bigsqcup_{n\geq 0} f_n)(r)(\omega) = 0 = (\bigsqcup_{n\geq 0} \mathscr{F}(f_n))(r)(\omega)$$
for all $\omega \in \Omega$.
3. Otherwise, $r \xrightarrow{\tau} \Delta$ for some $\Delta \in \mathscr{D}(R)$. Then we infer that, for any $\omega \in \Omega$,
$$\begin{array}{lcll} \mathscr{F}(\bigsqcup_{n\geq 0} f_n)(r)(\omega) & = & \mathrm{Exp}_\Delta(\bigsqcup_{n\geq 0} f_n)(\omega) & \text{by (3)} \\ & = & \sum_{r\in\lceil\Delta\rceil} \Delta(r) \cdot (\bigsqcup_{n\geq 0} f_n)(r)(\omega) & \\ & = & \sum_{r\in\lceil\Delta\rceil} \Delta(r) \cdot (\bigsqcup_{n\geq 0} f_n(r))(\omega) & \\ & = & \sum_{r\in\lceil\Delta\rceil} \bigsqcup_{n\geq 0} \Delta(r) \cdot f_n(r)(\omega) & \\ & = & \sum_{r\in\lceil\Delta\rceil} \lim_{n\to\infty} \Delta(r) \cdot f_n(r)(\omega) & \\ & = & \lim_{n\to\infty} \sum_{r\in\lceil\Delta\rceil} \Delta(r) \cdot f_n(r)(\omega) & \text{by Proposition 5.16} \\ & = & \bigsqcup_{n\geq 0} \sum_{r\in\lceil\Delta\rceil} \Delta(r) \cdot f_n(r)(\omega) & \\ & = & \bigsqcup_{n\geq 0} \mathrm{Exp}_\Delta(f_n)(\omega) & \\ & = & \bigsqcup_{n\geq 0} \mathscr{F}(f_n)(r)(\omega) & \\ & = & (\bigsqcup_{n\geq 0} \mathscr{F}(f_n))(r)(\omega)\;. & \end{array}$$
In the above reasoning, Proposition 5.16 can be applied because we can define the function $f : R \times \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ by letting $f(r,n) = \Delta(r) \cdot f_n(r)(\omega)$ and checking that f satisfies the three conditions in Proposition 5.16. If R is finite, we can extend it to a countable set $R' \supseteq R$ and require $f(r',n) = 0$ for all $r' \in R' \backslash R$ and $n \in \mathbb{N}$.

(a) f satisfies condition **C1**. For any $r \in R$ and $j_1, j_2 \in \mathbb{N}$, if $j_1 \leq j_2$ then $f_{j_1} \leq f_{j_2}$. It follows that
$$f(r, j_1) \;=\; \Delta(r) \cdot f_{j_1}(r)(\omega) \;\leq\; \Delta(r) \cdot f_{j_2}(r)(\omega) \;=\; f(r, j_2).$$

(b) f satisfies condition **C2**. For any $r \in R$, the sequence $\{\Delta(r) \cdot f_n(r)(\omega)\}_{n\geq 0}$ is nondecreasing and bounded by $\Delta(r)$. It follows that the limit $\lim_{n\to\infty} f(r,n)$ exists.

(c) f satisfies condition **C3**. For any $R'' \subseteq R$, the partial sum $\sum_{r\in R''} \lim_{n\to\infty} f(r,n)$ is bounded because
$$\sum_{r\in R''} \lim_{n\to\infty} f(r,n) = \sum_{r\in R''} \lim_{n\to\infty} \Delta(r) \cdot f_n(r)(\omega) \leq \sum_{r\in R''} \Delta(r) \leq \sum_{r\in R} \Delta(r) = 1. \qquad \square$$

The continuity of $\mathscr{F}$ implies that its fixed point $\mathbb{V}$ can be captured by a chain of approximants. The functions $\mathbb{V}^n$, $n \geq 0$ are defined by induction on n:

$$\begin{aligned} \mathbb{V}^0(r)(\omega) &= 0 \qquad \text{for all } r \in R \text{ and } \omega \in \Omega \\ \mathbb{V}^{n+1} &= \mathscr{F}(\mathbb{V}^n) \end{aligned}$$

Now $\mathbb{V} = \bigsqcup_{n\geq 0} \; \mathbb{V}^n$. This is used in the following result.

Lemma 5.19 Let Λ be a subdistribution in an ω-respecting deterministic pLTS $\langle R, \Omega, \rightarrow_R \rangle$. If $\Lambda \Longrightarrow\!\!\succ \Lambda'$ then $\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle}) = \mathrm{Exp}_{\Lambda'}(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle})$.

Proof: For simplicity let us write $\mathbb{V}(\Delta)$ for $\mathrm{Exp}_\Delta(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle})$ for any Δ. Since the pLTS is ω-respecting we know that $s \xrightarrow{\tau} \Delta$ implies $s \stackrel{\omega}{\not\longrightarrow}$ for any ω. Therefore, from the definition of the functional $\mathscr{F}$ we have that $s \xrightarrow{\tau} \Delta$ implies $\mathbb{V}^{n+1}(s) = \mathbb{V}^n(\Delta)$, whence by lifting and linearity we get:

$$\text{if } \Delta \xrightarrow{\tau} \Delta' \text{ then } \mathbb{V}^{n+1}(\Delta) = \mathbb{V}^n(\Delta') \text{ for all } n \geq 0.$$

Now suppose $\Lambda \Longrightarrow\!\!\succ \Lambda'$. Then

$$\Lambda = \Lambda_0, \qquad \Lambda_k = \Lambda_k^\times + \Lambda_k^\rightarrow, \qquad \Lambda_k^\rightarrow \xrightarrow{\tau} \Lambda_{k+1}, \qquad \Lambda' = \sum_{k=0}^{\infty} \Lambda_k^\times .$$

Using in the base case that $\mathbb{V}^0(\Delta)(\omega) = 0$ for each Δ, a straightforward induction on n yields, for all $\ell \geq 0$,

$$\mathbb{V}^n(\Lambda_\ell) \quad = \quad \sum_{k=0}^{n} \mathbb{V}^{n-k}(\Lambda_{\ell+k}^\times) \,. \tag{13}$$

Namely $\mathbb{V}^{n+1}(\Lambda_\ell) = \mathbb{V}^{n+1}(\Lambda_\ell^\times + \Lambda_\ell^\rightarrow) = \mathbb{V}^{n+1}(\Lambda_\ell^\times) + \mathbb{V}^{n+1}(\Lambda_\ell^\rightarrow) = \mathbb{V}^{n+1}(\Lambda_\ell^\times) + \mathbb{V}^n(\Lambda_{\ell+1}) \stackrel{\text{induction}}{=} \mathbb{V}^{n+1}(\Lambda_\ell^\times) + \sum_{k=0}^n \mathbb{V}^{n-k}(\Lambda_{\ell+1+k}^\times) = \mathbb{V}^{n+1}(\Lambda_\ell^\times) + \sum_{k=1}^{n+1} \mathbb{V}^{n+1-k}(\Lambda_{\ell+k}^\times) = \sum_{k=0}^{n+1} \mathbb{V}^{n+1-k}(\Lambda_{\ell+k}^\times)$.

Since $\Lambda_k^\times$ is stable, we have

$$\mathbb{V}^m(\Lambda_k^\times) = \mathbb{V}(\Lambda_k^\times) \qquad \text{for every } k, m \geq 0. \tag{14}$$

We conclude by reasoning

$$\begin{array}{lll} \mathbb{V}(\Lambda) & = \bigsqcup_{n\geq 0} \mathbb{V}^n(\Lambda) & \text{by continuity of } \mathscr{F} \\ & = \bigsqcup_{n\geq 0} \sum_{k=0}^n \mathbb{V}^{n-k}(\Lambda_k^\times) & \text{from (13) above, taking } \ell = 0 \\ & = \bigsqcup_{n\geq 0} \sum_{k=0}^n \mathbb{V}(\Lambda_k^\times) & \text{by (14)} \\ & = \bigsqcup_{n\geq 0} \mathbb{V}(\sum_{k=0}^n \Lambda_k^\times) & \text{by linearity of } \mathbb{V} \\ & = \mathbb{V}(\bigsqcup_{n\geq 0} \sum_{k=0}^n \Lambda_k^\times) & \text{by Lemma 5.17} \\ & = \mathbb{V}(\sum_{k=0}^\infty \Lambda_k^\times) & \\ & = \mathbb{V}(\Lambda') \,. & \end{array} \qquad \square$$

We are now ready to compare the two methods for calculating the set of outcomes associated with a subdistribution:

- using extreme derivatives and the reward function \$ from Definition 5.3
- using resolutions and the evaluation function $\mathbb{V}$ from page 6.

Theorem 5.20 In an ω-respecting pLTS $\langle S,\Omega,\rightarrow\rangle$, the following statements hold.

(a) If $\Phi \Longrightarrow\!\!\succ \Phi'$ then there is a resolution $\langle R,\Lambda,\rightarrow_R\rangle$ of Φ such that $\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle}) = \$\Phi'$.

(b) For any resolution $\langle R,\Lambda,\rightarrow_R\rangle$ of Φ, there exists a Φ' such that $\Phi \Longrightarrow\!\!\succ \Phi'$ and $\mathrm{Exp}_\Lambda(\mathbb{V}_{\langle R,\Omega,\rightarrow_R\rangle}) = \$\Phi'$.

Proof: Suppose $\Phi \Longrightarrow\!\!\succ \Phi'$. By Proposition 5.12, there is a resolution $\langle R,\Lambda,\rightarrow_R\rangle$ of Φ with resolving function f and a subdistribution Λ' such that $\Lambda \Longrightarrow\!\!\succ \Lambda'$ and $\Phi' = \mathrm{Img}_f(\Lambda')$. By Lemma 5.19, we have

$$\mathrm{Exp}_\Lambda(\mathbb{V}) = \mathrm{Exp}_{\Lambda'}(\mathbb{V}). \tag{15}$$

Since Λ' is an extreme derivative, all the states s in its support are stable, so $\mathbb{V}(s)(\omega) = 0$ if $s \stackrel{\omega}{\not\longrightarrow}$, for all $\omega \in \Omega$. Hence

$$\mathrm{Exp}_{\Lambda'}(\mathbb{V})(\omega) = \sum_{s\in\lceil\Lambda'\rceil} \Lambda'(s)\cdot\mathbb{V}(s)(\omega) = \sum_{s\in\lceil\Lambda'\rceil,\ s\xrightarrow{\omega}} \Lambda'(s) = \$\Lambda'(\omega)\ . \tag{16}$$

Furthermore, for all $t \in \lceil\Phi'\rceil$, $\Phi'(t) = \mathrm{Img}_f(\Lambda')(t) = \sum_{f(s)=t}\Lambda'(s)$, so, for all $\omega\in\Omega$,

$$\$\Phi'(\omega) = \sum_{t\in\lceil\Phi'\rceil,\ t\xrightarrow{\omega}} \Phi'(t) = \sum_{t\in\lceil\Phi'\rceil,\ t\xrightarrow{\omega}} \mathrm{Img}_f(\Lambda')(t) = \sum_{t\in\lceil\Phi'\rceil,\ t\xrightarrow{\omega}} \textstyle\sum_{f(s)=t}\Lambda'(s) = \displaystyle\sum_{s\in\lceil\Lambda'\rceil,\ f(s)\xrightarrow{\omega}} \Lambda'(s) = \$\Lambda'(\omega)\ ,$$

where in the last step we the use the property of resolutions that $f(s) \xrightarrow{\omega}$ iff $s \xrightarrow{\omega}$. Combining this with (15) and (16) yields that $\mathrm{Exp}_\Lambda(\mathbb{V}) = \$\Phi'$.

To prove part (b), suppose that $\langle R,\Lambda,\rightarrow_R\rangle$ is a resolution of Φ with resolving function f, so that $\Phi = \mathrm{Img}_f(\Lambda)$. We know from Lemma 5.2 that there exists a (unique) subdistribution Λ' such that $\Lambda \Longrightarrow\!\!\succ \Lambda'$. By Proposition 5.13 we have that $\Phi \Longrightarrow\!\!\succ \mathrm{Img}_f(\Lambda')$. The same arguments as in the other direction show that $\mathrm{Exp}_\Lambda(\mathbb{V}) = \$(\mathrm{Img}_f(\Lambda'))$. □

A direct consequence of the above theorem is that $\mathcal{V}(\Phi) = \mathcal{A}(\Phi)$ for any subdistribution Φ in an ω-respecting pLTS $\langle S,\Omega,\rightarrow\rangle$. This implies that $\mathcal{V}([\Phi]) = \mathcal{A}^{\mathrm{p}}(\Phi)$ for any subdistribution Φ in a pLTS $\langle S,\Omega,\rightarrow\rangle$. This, in turn, together with Proposition 5.11, implies the following result.

Corollary 5.21 For any subdistribution Φ in a pLTS $\langle S,\Omega,\rightarrow\rangle$ we have that $\mathcal{V}([\Phi]) = \mathcal{A}(\Phi)$. □

6 Agreement of nonnegative- and real-reward must testing

In this section we prove the agreement of $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$ with $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$ for finitary convergent processes, by using failure simulation [3], recalled in Definition 4.6, as a stepping stone.

Because we prune our pLTSs before extracting values from them, we will be concerned mainly with ω-respecting structures. Moreover, we require the pLTSs to be *convergent* in the sense that there is no wholly divergent state s, i.e. with $s \Longrightarrow \varepsilon$. It follows from Theorem 8 in [3], in combination with Lemma 4.4(iii), that on a finitary convergent pLTS, if $\Delta \Longrightarrow \Delta'$ with Δ a full distribution, then Δ' is a full distribution.

Lemma 6.1 Let Δ and Γ be full distributions in an ω-respecting finitary convergent pLTS $\langle S,\Omega,\rightarrow\rangle$. If distribution Γ is stable and $\Gamma \,\overline{\lhd_{FS}}\, \Delta$, then $\$\Gamma \in \mathcal{V}(\Delta)$.

Proof: We first show that if s is stable and $s \lhd_{FS} \Delta$ with Δ a full distribution, then $\$\overline{s} \in \mathcal{V}(\Delta)$. Since s is stable, we have only two cases:

(i) $s \not\rightarrow$ Here $\$\overline{s} = \overrightarrow{0}$, where $\overrightarrow{0}(\omega) = 0$ for all $\omega \in \Omega$. Since $s \lhd_{FS} \Delta$ we have $\Delta \Longrightarrow \Delta'$ with $\Delta' \not\rightarrow$, whence in fact $\Delta \Longrightarrow\!\!\succ \Delta'$ and $\$\Delta' = \overrightarrow{0}$. Thus $\$\overline{s} = \overrightarrow{0} \in \mathcal{V}(\Delta)$.

(ii) $s \xrightarrow{\omega} \Gamma'$ for some Γ' Here $\$\overline{s} = \overrightarrow{\omega}$, and since $s \lhd_{FS} \Delta$ we have $\Delta \Longrightarrow \Delta' \xrightarrow{\omega}$. As remarked above, also Δ' is a full distribution. Hence $\$\Delta' = \overrightarrow{\omega}$. Because the pLTS is ω-respecting, in fact $\Delta \Longrightarrow\!\!\succ \Delta'$ and so again $\$\overline{s} = \overrightarrow{\omega} \in \mathcal{V}(\Delta)$.

Now for the general case we suppose $\Gamma \mathrel{\overline{\lhd_{FS}}} \Delta$. By Lemma 4.2 there is an index set I and states s_i, subdistributions Δ_i and probabilities p_i for $i \in I$, with $\sum_{i \in I} p_i \leq 1$, such that

$$\Gamma = \textstyle\sum_{i \in I} p_i \cdot \overline{s_i}, \qquad s_i \lhd_{FS} \Delta_i \text{ for each } i \in I \qquad \text{and} \qquad \Delta = \textstyle\sum_{i \in I} p_i \cdot \Delta_i .$$

Since Δ is full, $\sum_{i \in I} p_i = 1$ and the Δ_i are full distributions. Since Γ is stable, each state s_i is stable. From above we have that $\$\overline{s_i} \in \mathcal{V}(\Delta_i)$ for all $i \in I$, and so $\$\Gamma = \sum_{i \in I} p_i \cdot \$\overline{s_i} \in \sum_{i \in I} p_i \cdot \mathcal{V}(\Delta_i) = \mathcal{V}(\Delta)$, using Lemma 5.4. □

Lemma 6.2 Let Δ and Γ be full distributions in an ω-respecting finitary convergent pLTS $\langle S, \Omega, \rightarrow \rangle$. Then $\Delta \sqsubseteq_{FS} \Gamma$ implies $\mathcal{V}(\Delta) \supseteq \mathcal{V}(\Gamma)$.

Proof: Let $\Gamma, \Delta \in \mathcal{D}(S)$. We first claim that

(i) If $\Delta \Longrightarrow \Delta'$ then $\mathcal{V}(\Delta') \subseteq \mathcal{V}(\Delta)$.

(ii) If $\Gamma \mathrel{\overline{\lhd_{FS}}} \Delta$, then we have $\mathcal{V}(\Gamma) \subseteq \mathcal{V}(\Delta)$.

The first claim holds because if $\Delta' \Longrightarrow\!\!\succ \Delta''$ then $\Delta \Longrightarrow \Delta' \Longrightarrow\!\!\succ \Delta''$, i.e. every extreme derivative of Δ' is also an extreme derivative of Δ. For the second claim, we assume $\Gamma \mathrel{\overline{\lhd_{FS}}} \Delta$. For any $\Gamma \Longrightarrow\!\!\succ \Gamma'$ Lemma 4.7 gives a matching transition $\Delta \Longrightarrow \Delta'$ such that $\Gamma' \mathrel{\overline{\lhd_{FS}}} \Delta'$. By definition Γ' is stable and since $\langle S, \Omega, \rightarrow \rangle$ is finitary and convergent Δ' and Γ' must be full. It follows from Lemma 6.1 and Claim (i) that $\$\Gamma' \in \mathcal{V}(\Delta') \subseteq \mathcal{V}(\Delta)$. Consequently, we obtain $\mathcal{V}(\Gamma) \subseteq \mathcal{V}(\Delta)$.

Now suppose $\Delta \sqsubseteq_{FS} \Gamma$. By definition there exists some Δ' such that $\Delta \Longrightarrow \Delta'$ and $\Gamma \mathrel{\overline{\lhd_{FS}}} \Delta'$. By the above two claims we obtain $\mathcal{V}(\Gamma) \subseteq \mathcal{V}(\Delta') \subseteq \mathcal{V}(\Delta)$. □

This lemma shows that the failure-simulation preorder is a very strong relation in the sense that if Δ is related to Γ by the failure-simulation preorder then the set of outcomes generated by Δ includes the set of outcomes given by Γ. It is mainly due to this strong property that we can show that the failure-simulation preorder is sound for the real-reward must-testing preorder. Convergence is a crucial condition in this lemma.

Theorem 6.3 For any finitary convergent processes Δ and Γ, if $\Delta \sqsubseteq_{FS} \Gamma$ then we have that $\Delta \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Gamma$.

Proof: We reason as follows.

$$\begin{array}{lll}
 & \Delta \sqsubseteq_{FS} \Gamma & \\
\text{implies} & [\Theta \| \Delta] \sqsubseteq_{FS} [\Theta \| \Gamma] & \text{Lemma 5.10, for any } \Omega\text{-test } \Theta \\
\text{implies} & \mathcal{V}([\Theta \| \Delta]) \supseteq \mathcal{V}([\Theta \| \Gamma]) & [\cdot] \text{ is } \omega\text{-respecting; Lemma 6.2} \\
\text{iff} & \mathcal{A}(\Theta, \Delta) \supseteq \mathcal{A}(\Theta, \Gamma) & \text{Corollary 5.21} \\
\text{implies} & h \cdot \mathcal{A}(\Theta, \Delta) \supseteq h \cdot \mathcal{A}(\Theta, \Gamma) \quad \text{for any } h \in [-1,1]^{\Omega} & \\
\text{implies} & \bigsqcap h \cdot \mathcal{A}(\Theta, \Delta) \leq \bigsqcap h \cdot \mathcal{A}(\Theta, \Gamma) \quad \text{for any } h \in [-1,1]^{\Omega} & \\
\text{iff} & \Delta \sqsubseteq^{\Omega}_{\mathbf{rr}\mathrm{must}} \Gamma . &
\end{array}$$

Note that in the second line above, both $[\Theta\|\Delta]$ and $[\Theta\|\Gamma]$ are convergent, since for any convergent process Ξ and finite process Θ, by induction on the structure of Θ, it can be shown that the composition $\Theta\|\Xi$ is also convergent. Furthermore, since processes Δ,Γ and tests Θ are defined to be full distributions, also $[\Theta\|\Delta]$ and $[\Theta\|\Gamma]$ are full. □

The proof of the above theorem is subtle. The failure-simulation preorder is defined via weak derivations (cf. Definition 4.6), while the reward must-testing preorder is defined in terms of resolutions (cf. Definition 3.5). Fortunately, we have shown in Corollary 5.21 that we can just as well characterise the reward must-testing preorder in terms of weak derivations. Based on this observation, the proof was carried out by exploiting Lemmas 5.10 and 6.2.

This result does not extend to divergent processes. One witness example is given in Figure 1. A simpler example is as follows. Let Δ be a process that diverges, by performing a τ-loop only, and let Γ be a process that merely performs a single action a. It holds that $\Delta \sqsubseteq_{FS} \Gamma$ because $\Delta \Longrightarrow \varepsilon$ and the empty subdistribution can failure-simulate any processes. However, if we apply the test $\bar{t}$ from Example 3.2 again, and the reward tuple h with $h(\omega) = -1$, then

$$\begin{array}{lllllllll}
\prod h\cdot\mathscr{A}(\bar{t},\Delta) & = & \prod h\cdot\mathscr{V}([\bar{t}\|\Delta]) & = & \prod h\cdot\{\$\varepsilon\} & = & \prod\{0\} & = & 0\\
\prod h\cdot\mathscr{A}(\bar{t},\Gamma) & = & \prod h\cdot\mathscr{V}([\bar{t}\|\Gamma]) & = & \prod h\cdot\{\overrightarrow{\omega}\} & = & \prod\{-1\} & = & -1
\end{array}$$

As $\prod h\cdot\mathscr{A}(\bar{t},\Delta) \not\leq \prod h\cdot\mathscr{A}(\bar{t},\Gamma)$, we see that $\Delta \not\sqsubseteq_{\mathbf{rr}\mathrm{must}} \Gamma$. Since $\mathscr{V}([\bar{t}\|\Gamma]) = \{\overrightarrow{\omega}\}$ but $\overrightarrow{\omega} \notin \mathscr{V}([\bar{t}\|\Delta])$, this also is a counterexample against an extension of Lemma 6.2 with divergence.

Finally, by combining Theorems 3.6(ii) and 4.8(ii), together with Theorem 6.3, we obtain the main result of the paper which states that, in the absence of divergence, nonnegative-reward must testing is as discriminating as real-reward must testing.

Theorem 6.4 For any finitary convergent processes Δ and Γ, it holds that $\Delta \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Gamma$ if and only if $\Delta \sqsubseteq_{\mathbf{nr}\mathrm{must}} \Gamma$.

Proof: The "only if" direction is obvious (cf. Definition 3.5). For the "if" direction, suppose Δ and Γ are finitary convergent processes. We reason as follows.

	$\Delta \sqsubseteq^{\Omega}_{\mathbf{nr}\mathrm{must}} \Gamma$	
iff	$\Delta \sqsubseteq^{\Omega}_{\mathrm{pmust}} \Gamma$	Theorem 3.6(ii)
iff	$\Delta \sqsubseteq_{FS} \Gamma$	Theorem 4.8(ii)
implies	$\Delta \sqsubseteq^{\Omega}_{\mathbf{rr}\mathrm{must}} \Gamma$.	Theorem 6.3 □

7 Discussion

Below we give a characterisation of $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$ in terms of the set inclusion relation between testing outcome sets. As a similar characterisation for $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$ does in general not hold for finitary (non-convergent) processes, hopefully this gives some indication of the subtle difference between $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$ and $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$, and we see more clearly why our proof of Theorem 6.4 involves the failure simulation preorder.

Theorem 7.1 Let Δ and Γ be any finitary processes. Then $\Delta \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Gamma$ if and only if $\mathscr{A}(\Theta,\Delta) \supseteq \mathscr{A}(\Theta,\Gamma)$ for any Ω-test Θ.

Proof: ($\Leftarrow$) Let Θ be any Ω-test and $h \in [-1,1]^{\Omega}$ be any real-reward tuple. Suppose $\mathscr{A}(\Theta,\Delta) \supseteq \mathscr{A}(\Theta,\Gamma)$. It is obvious that $h\cdot\mathscr{A}(\Theta,\Delta) \supseteq h\cdot\mathscr{A}(\Theta,\Gamma)$, from which it easily follows that

$$\prod h\cdot\mathscr{A}(\Theta,\Delta) \;\leq\; \prod h\cdot\mathscr{A}(\Theta,\Gamma).$$

As this holds for an arbitrary real-reward tuple h, we see that $\Delta \sqsubseteq_{\mathbf{rr}\text{must}} \Gamma$.

($\Rightarrow$) Suppose for a contradiction that there is some Ω-test Θ with $\mathscr{A}(\Theta,\Delta) \not\supseteq \mathscr{A}(\Theta,\Gamma)$. Then there exists some outcome $o \in \mathscr{A}(\Theta,\Gamma)$ lying outside $\mathscr{A}(\Theta,\Delta)$, i.e.

$$o \notin \mathscr{A}(\Theta,\Delta). \tag{17}$$

Since Θ is finite, it contains only finitely many elements of Ω, so that we may assume wlog that Ω is finite. Since Δ and Θ are finitary, it is easy to see that the pruned composition $[\Delta\|\Theta]$ is also finitary. By Theorem 1/Corollary 1 in [3], the set $\{\Phi \mid [\Delta\|\Theta] \Longrightarrow \Phi\}$ is convex and compact. With an analogous proof, it can be shown that so is the set $\{\Phi \mid [\Delta\|\Theta] \Longrightarrow\!\!\succ \Phi\}$. It follows that the set

$$\{\$\Phi \mid [\Delta\|\Theta] \Longrightarrow\!\!\succ \Phi\}$$

i.e. $\mathscr{V}([\Theta\|\Delta])$, is also convex and compact. By Corollary 5.21 the set $\mathscr{A}(\Theta,\Delta)$ is thus convex and compact. Combining this with (17), and using the Separation Hyperplane Lemma [7, 12], we infer the existence of some hyperplane whose normal is $h \in \mathbb{R}^\Omega$ such that $h \cdot o' > h \cdot o$ for all $o' \in \mathscr{A}(\Theta,\Delta)$. By scaling h, we obtain without loss of generality that $h \in [-1,1]^\Omega$. It follows that

$$\textstyle\bigsqcap h \cdot \mathscr{A}(\Theta,\Delta) \;>\; h \cdot o \;\geq\; \bigsqcap h \cdot \mathscr{A}(\Theta,\Gamma)$$

which is a contradiction to the assumption that $\Delta \sqsubseteq_{\mathbf{rr}\text{must}} \Gamma$. □

Note that in the above proof the normal of the separating hyperplane belongs to $[-1,1]^\Omega$ rather than $[0,1]^\Omega$. So we cannot repeat the above proof for $\sqsubseteq_{\mathbf{nr}\text{must}}$. In general, we do not have that $\Delta \sqsubseteq_{\mathbf{nr}\text{must}} \Gamma$ implies $\mathscr{A}(\Theta,\Delta) \supseteq \mathscr{A}(\Theta,\Gamma)$ for any Ω-test Θ and for arbitrary finitary processes Δ and Γ, that is finitary processes which might not be convergent. However, when we restrict ourselves to finitary convergent processes, this property does indeed hold, as can be seen from the first four lines in the proof of Theorem 6.3. Note that in that proof there is an essential use of the failure simulation preorder; in particular the pleasing property stated in Lemma 6.2. Even for finitary convergent processes we cannot give a direct and simple proof of that property for $\sqsubseteq_{\mathbf{nr}\text{must}}$, analogous to that of Theorem 7.1.

8 Conclusion

We have studied a notion of real-reward testing which extends the traditional nonnegative-reward testing with negative rewards. It turned out that the real-reward may preorder is the inverse of the real-reward must preorder, and vice versa. More interestingly, for finitary convergent processes, the real-reward must testing preorder coincides with the nonnegative-reward testing preorder. In order to prove this result, we have capitalised on a characterisation of nonnegative-reward testing in terms of a derivation based simulation preorder. Relating derivations to resolutions, on which the testing theories are based, involved proving some analytic properties such as the continuity of a function for calculating testing outcomes.

Although for finitary convergent processes real-reward must testing is no more powerful than nonnegative-reward must testing, the same does not hold for may testing. This is immediate from our result that (the inverse of) real-reward may testing is as powerful as real-reward must testing, that is known not to hold for nonnegative-reward may- and must testing. For finitary processes we know from [3] that $\sqsubseteq_{\mathbf{nr}\text{may}}$ and $\sqsubseteq_{\mathbf{nr}\text{must}}$ correspond to the simulation and failure simulation preorder respectively, and without divergence the latter is strictly more discriminating than the former.

References

[1] R. De Nicola & M. Hennessy (1984): *Testing equivalences for processes*. Theoretical Computer Science 34, pp. 83–133, doi:10.1016/0304-3975(84)90113-0.

[2] Y. Deng, R.J. van Glabbeek, M. Hennessy & C.C. Morgan (2008): *Characterising testing preorders for finite probabilistic processes*. Logical Methods in Computer Science 4(4):4, doi:10.2168/LMCS-4(4:4)2008.

[3] Y. Deng, R.J. van Glabbeek, M. Hennessy & C.C. Morgan (2009): *Testing finitary probabilistic processes*. In: Proc. *CONCUR'09*, LNCS 5710, Springer, pp. 274–288, doi:10.1007/978-3-642-04081-8_19.

[4] Y. Deng, R.J. van Glabbeek, M. Hennessy & C.C. Morgan (2009): *Testing finitary probabilistic processes.* Full version of [3]. Available at `http://www.cse.unsw.edu.au/~rvg/pub/finitary.pdf`.

[5] Y. Deng, R.J. van Glabbeek, M. Hennessy & C.C. Morgan (2011): *Real Reward Testing for Probabilistic Processes*. In: Proc. *QAPL'11*. *EPTCS* 57, pp. 61–73, doi:10.4204/EPTCS.57.5.

[6] Y. Deng, R.J. van Glabbeek, M. Hennessy, C.C. Morgan & C. Zhang (2007): *Remarks on Testing Probabilistic Processes*. *ENTCS* 172, pp. 359–397, doi:10.1016/j.entcs.2007.02.013.

[7] Y. Deng, R.J. van Glabbeek, C.C. Morgan & C. Zhang (2007): *Scalar Outcomes Suffice for Finitary Probabilistic Testing*. In: Proc. *ESOP'07*, LNCS 4421, Springer, pp. 363–368, doi:10.1007/978-3-540-71316-6_25.

[8] R.J. van Glabbeek (1993): *The Linear Time – Branching Time Spectrum II; The semantics of sequential systems with silent moves (extended abstract)*. In: Proc. *CONCUR'93*. LNCS 751, Springer, pp. 66–81, doi:10.1007/3-540-57208-2_6.

[9] M. Hennessy (1988): *An Algebraic Theory of Processes*. MIT Press.

[10] B. Jonsson, C. Ho-Stuart & Wang Yi (1994): *Testing and Refinement for Nondeterministic and Probabilistic Processes*. In: Proc. *FTRTFT'94*, LNCS 863, Springer, pp. 418–430, doi:10.1007/3-540-58468-4_176.

[11] D. Kozen (1985): *A Probabilistic PDL*. *JCSS* 30(2), pp. 162–178, doi:10.1016/0022-0000(85)90012-1.

[12] J. Matousek (2002): *Lectures on Discrete Geometry*. Springer.

[13] R. Milner (1989): *Communication and Concurrency*. Prentice-Hall.

[14] M.L. Puterman (1994): *Markov Decision Processes*. Wiley, doi:10.1002/9780470316887.

[15] J.J.M.M. Rutten, M.Kwiatkowska, G. Norman & D. Parker (2004): *Mathematical Techniques for Analyzing Concurrent and Probabilistic Systems,* P. Panangaden and F. van Breugel (eds.). *CRM Monograph Series* 23, American Mathematical Society.

[16] R. Segala (1995): *Modeling and Verification of Randomized Distributed Real-Time Systems*. Ph.D. thesis, MIT.

[17] R. Segala (1996): *Testing Probabilistic Automata*. In: Proceedings *CONCUR'96*, LNCS 1119, Springer, pp. 299–314, doi:10.1007/3-540-61604-7_62.

[18] Wang Yi & K.G. Larsen (1992): *Testing Probabilistic and Nondeterministic Processes*. In: Proc. *PSTV'92. IFIP Transactions* C-8, North-Holland, pp. 47–61.

[19] Glynn Winskel (1993): *The Formal Semantics of Programming Languages: An Introduction*. The MIT Press.