Models for Polymorphic Lambek Calculus

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Abstract. An adaptation of *string-semantic* models to the Polymorphic Lambek Calculus is studied. We note that if quantifiers range over arbitrary sets of strings, the polymorphic calculus is incomplete. The semantics is refined so that quantifiers range over sets of strings that are the interpretation of categories, and we prove a completeness result. In addition some other models are considered and there are speculations on the relevance of model theory to linguistics.

1 Monomorphic Grammars and Models

In this paper we investigate models of the polymorphic Lambek calculus, an extension of the Lambek calculus with quantification of category variables. Several notions of model for the quantifier-free calculus are considered and extended to the polymorphic case. The Lambek calculus, and its polymorphic extension have their foremost application in linguistic analysis. By way of motivation for the later results, this section discuss the significance of model theory for linguistics. The section is somewhat speculative; the two research streams, linguistic and model-theoretic, are substantial, but concerning their confluence little has been said. Besides the material introducing the calculi and models, the later sections can be read independently of this one.

1.1 Calculi and grammars

First some preliminaries. Categorial grammar formalisms can generally be seen as logics. Since the connection was made most strikingly by Lambek, and especially in the last decade, explorations have been made of a number of parameters of variation in these logics. One parameter is *language* of the logic. We will use $\mathcal{L}$, followed by a series of connectives drawn from the set $\{\bullet, \mathbf{1}, /, \backslash, \forall, \exists\}$ to specify a particular language, and refer to elements of such a language as categories or sometimes formulae.

Both the Adjukiewicz/Bar-Hillel [2] [3] and Lambek [16] categorial grammar formalisms are based on the same language $\mathcal{L}(/, \backslash)$. They differ with respect to the sequents over $\mathcal{L}(/, \backslash)$ that they accept. The sets of sequents with which we will be concerned will be defined by calculi based on the rules in Figures 1.

These rules derive intuitionistic sequents from intuitionistic sequents, where an intuitionistic sequent is an antecedent sequence of formulae, followed by '$\Rightarrow$', and then a single formula. Concerning the notation we point out that w, x, y range over formulae, U, V, T range over sequences of formulae. $x[y/Z]$ stands for

$$x \Rightarrow x$$

$$\frac{U, y, V \Rightarrow w \quad T \Rightarrow x}{U, y/x, T, V \Rightarrow w}/\mathrm{L} \qquad \frac{T, x \Rightarrow y}{T \Rightarrow y/x}/\mathrm{R}$$

$$\frac{T \Rightarrow x \quad U, y, V \Rightarrow w}{U, T, x\backslash y, V \Rightarrow w}\backslash\mathrm{L} \qquad \frac{x, T \Rightarrow y}{T \Rightarrow x\backslash y}\backslash\mathrm{R}$$

$$\frac{U, x, y, V \Rightarrow w}{U, x{\bullet}y, V \Rightarrow w}\bullet\mathrm{L} \qquad \frac{T_1 \Rightarrow x \quad T_2 \Rightarrow y}{T_1, T_2 \Rightarrow x{\bullet}y}\bullet\mathrm{R}$$

$$\frac{U, x[y/Z], V \Rightarrow w}{U, \forall Z.x, V \Rightarrow w}\forall\mathrm{L} \qquad \frac{T \Rightarrow x}{T \Rightarrow \forall X.x[X/Z]}\forall\mathrm{R},\ Z!$$

$$\frac{U, x, V \Rightarrow w}{U, \exists X.x[X/Z], V \Rightarrow w}\exists\mathrm{L},\ Z! \qquad \frac{T \Rightarrow x[y/X]}{T \Rightarrow \exists X.x}\exists\mathrm{R}$$

$$\frac{U, V \Rightarrow w}{U, \mathbf{1}, V \Rightarrow w}\mathbf{1}\,\mathrm{L} \qquad \Rightarrow \mathbf{1}$$

$$\frac{T_1, T_2 \Rightarrow w}{T_1, x, T_2 \Rightarrow w}\mathrm{Weak} \qquad \frac{T_1, x, x, T_2 \Rightarrow w}{T_1, x, T_2 \Rightarrow w}\mathrm{Contr} \qquad \frac{\pi(T) \Rightarrow w}{T \Rightarrow w}\mathrm{Perm}$$

Fig. 1. Identity axiom, and rules for $/,\backslash,\bullet,\forall,\exists,\mathbf{1}$,Structural Rules

the substitution of y for Z in x, defined to include a change of bound variable to avoid accidental capture. The side condition 'Z!' is that Z is not free below the line, and X is not free in $QZ.x$, this latter part allowing $QX.x[X/Z]$ to be an alphabetic variant of $QZ.x$.

The Adjukiewicz/Bar-Hillel formalism is based on the calculus obtained by taking the identity axiom scheme, together with (/L) and (\L), whilst the Lambek formalism takes additionally the (/R) and (\R) rules. In order to have systematic names for other choices, we use $\mathrm{L}^{\langle c_i \rangle_{1\leq i\leq n}}$ for (the set of sequents derivable using) the calculus obtained by taking the identity axiom scheme, together with the rules associated with each of the connectives c_i, with in the case of $\forall$L and $\exists$R, the obvious proviso that formula y substituted for the bound variable be drawn from the appropriate language: $\mathcal{L}(\langle c_i \rangle_{1\leq i\leq n})$. The calculi including the second-order quantifiers we will refer to as *polymorphic*. In none of the calculi $\mathrm{L}^{\langle c_i \rangle_{1\leq i\leq n}}$ are the structural rules of Figure 1 admissible, and hence all may be referred to as non-commutative, linear logics. For any of the calculi that may be defined in this way, the derivability of the following Cut rule can be shown (by an inductive argument on the sizes of the premise proofs of a Cut[1])

$$\frac{T \Rightarrow x \qquad U, x, V \Rightarrow w}{U, T, V \Rightarrow w}\text{Cut}$$

Where L is a sequent calculus for a categorial language $\mathcal{L}$, there is a standard conception of how this may be put to work to define a grammar[2]. First let us call a relation $R \subseteq V^+ \times \mathcal{L}$, L- *closed* if whenever $(s_1, x_1) \in R, \ldots, (s_n, x_n) \in R$, and $\text{L} \vdash x_1, \ldots, x_n \Rightarrow y$ then $(s_1 \ldots s_n, y) \in R$ (where V^+ is the closure of V under concatenation).

Definition 1 (L-grammar) *Where V is a vocabulary, $\mathcal{L}$ some categorial language,* L *a sequent calculus for this language, and G_0 some finite relation on $V \times \mathcal{L}$ (known as the lexicon), the* L-grammar *G is the least relation on $V^+ \times \mathcal{L}$, satisfying i. $G_0 \subseteq G$ and ii. G is* L*-closed.*

For any relation $R \subseteq A \times B$, we use $(x)_R$, for $\{a \in A \mid (a, x) \in R\}$. Then choosing some distinguished category, x_0, the set $(x_0)_G$ can be seen as the *language generated by G.*

Example 1: Adjukiewicz grammar : let $V = \{a, b, c\}$, $\mathcal{L} = \{/, \backslash\}$, $G_0 = \{(a, \text{np}), (b, \text{s/s}), (b, \text{np}\backslash\text{np}), (c, \text{np}\backslash\text{s})\}$, L = identity axiom + (/L) + (\L). Then the values of $(\cdot)_G$ on some of the categories are:

$$\begin{aligned}
(\text{np})_G &= \{ab^p \mid p \geq 0\}\\
(\text{s})_G &= \{b^p a b^q c \mid p \geq 0,\ q \geq 0\}\\
(\text{np}\backslash\text{np})_G &= \{b\}\\
(\text{s/s})_G &= \{b\}\\
(\text{np}\backslash\text{s})_G &= \{c\}
\end{aligned}$$

Example 2: Lambek grammar : as above, but with $\text{L} = \text{L}^{/,\backslash}$. Then the values of $(\cdot)$ on some of the categories are:

$$\begin{aligned}
(\text{np})_G &= \{ab^p \mid p \geq 0\}\\
(\text{s})_G &= \{b^p a b^q c \mid p \geq 0,\ q \geq 0\}\\
(\text{np}\backslash\text{np})_G &= \{b^p \mid p \geq 1\}\\
(\text{s/s})_G &= \{b^p \mid p \geq 1\}\\
(\text{np}\backslash\text{s})_G &= \{b^p c \mid p \geq 0\}
\end{aligned}$$

1.2 String semantic models

An L-grammar is a relation between V^+ and $\mathcal{L}$, where $\mathcal{L}$ is some categorial language. In the case of $\mathcal{L}(/, \backslash)$, such relations occur also in the so-called *string-semantic* models of $\mathcal{L}(/, \backslash)$, albeit in the form of functions $[\![\cdot]\!] : \mathcal{L} \rightarrow 2^{V^+}$: a

[1] Lambek's proof of Cut elimination for $\text{L}^{/,\bullet,\backslash}$ by induction on the complexity of the Cut formula, does not work for the polymorphic calculi. The absence of the contraction rule, however, allows a similarly simple proof to be given by induction on proof size. See [[12]].

[2] Due to Cut Elimination, restricting the s_i in the definition to members of V does not change G

category x receives an interpretation $[\![x]\!] \subseteq V^+$, subject to constraints generated by the connective structure of x

Definition 2 (String semantic model of $\mathcal{L}(/,\backslash)$) *$\langle S, \bullet, [\![]\!]\rangle$ is a string-semantic model if $\langle S, \bullet\rangle$ is a free semigroup and $[\![]\!]$ maps categories to subsets of S according to*
1. $[\![x/y]\!] = \{a \in S : \forall b \in [\![y]\!], a\bullet b \in [\![x]\!]\}$
2. $[\![y\backslash x]\!] = \{a \in S : \forall b \in [\![y]\!], b\bullet a \in [\![x]\!]\}$

Any such model is uniquely determined by the values of the basic categories, so that one can alternatively define the models by an interpretation for the basic categories, to be extended according to the above two conditions. We have made one departure from [4] in the above definition, omitting a requirement that S be the closure under $\bullet$ of some *finite* vocabulary V. The class of all string-semantic models we will refer to as S^{inf}. The corresponding class with finitely generated S, we will refer to as S [3].

Example 3: String Semantic Model : the values of $[\![\cdot]\!]$ on the basic categories are given first, followed by implied values on some of the categories:

$$
\begin{aligned}
[\![\mathrm{np}]\!] &= \{ab^p \mid p \geq 0\} \\
[\![\mathrm{s}]\!] &= \{b^p a b^q c \mid p \geq 0,\ q \geq 0\} \\
[\![\mathrm{np}\backslash\mathrm{np}]\!] &= \{b^p \mid p \geq 1\} \\
[\![\mathrm{s}/\mathrm{s}]\!] &= \{b^p \mid p \geq 1\} \\
[\![\mathrm{np}\backslash\mathrm{s}]\!] &= \{b^p c \mid p \geq 0\}
\end{aligned}
$$

The Lambek calculus generates exactly the inclusions true in all such models [4]. More exactly, if we say a sequent is *satisfied by a model*, $M \models x_1, \ldots, x_n \Rightarrow y$ iff $[\![x_1]\!]\bullet \ldots \bullet[\![x_n]\!] \subseteq [\![y]\!]$, then

Theorem 1 (Buszkoswki 1982). $\mathrm{L}^{/,\backslash} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all models* $M \in S$*,* $M \models x_1, \ldots, x_n \Rightarrow y$*.*

There is a harder to prove version of this [18] for the language $\mathcal{L}(/,\backslash,\bullet)$ and the calculus $\mathrm{L}^{/,\backslash,\bullet}$, where the conditions on $[\![\cdot]\!]$ are extended with:

$$[\![x\bullet y]\!] = \{a \in S : \exists b_1 \in [\![x]\!], \exists b_2 \in [\![y]\!], a = b_1\bullet b_2\}$$

Theorem 2 (Pentus 1993). $\mathrm{L}^{/,\backslash,\bullet} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all models* $M \in S$*,* $M \models x_1, \ldots, x_n \Rightarrow y$*.*

1.3 Other models

The string-semantic models are very concrete, and *construct* the value of a category as a set of strings. A somewhat more abstract semantics has also been investigated, which simply assumes the existence of objects which may serve as the values of categories. These are the *residuated semi-group* models.

[3] Thus S denotes the same class of models as it does in [4].

Definition 3 (Residuated Semigroup) $\langle M, \bullet, /, \backslash \rangle$ *is a residuated semi group if M is closed under* $\bullet$, $/$, $\backslash$, *the operation* $\bullet$ *is associative, M is partially ordered, and the operations relate to the ordering in the following way:*

$$a \leq c/b \text{ iff } a\bullet b \leq c \text{ iff } b \leq c\backslash a$$

Following [5], we will refer to the class of all residuated semi-groups as RES. Where C varies over connectives, and $\mathbf{C}$ over the corresponding operation, we define a RES-model as follows:

Definition 4 (RES model) $\langle M, \bullet, /, \backslash, [\![]\!] \rangle$ *is a* RES*-model if:* $\langle M, \bullet, /, \backslash \rangle \in RES$, *and where* C *is '/','\','•'* $[\![C(x,y)]\!] = \mathbf{C}([\![x]\!],[\![y]\!])$.

If we say a sequent is satisfied by a model, $\langle R, [\![]\!] \rangle \models x_1, \ldots, x_n \Rightarrow y$ iff $[\![x_1]\!]\bullet \ldots \bullet[\![x_n]\!] \leq [\![y]\!]$ then

Theorem 3 (Buszkowski 1986). $\mathrm{L}^{(/,\backslash,\bullet)} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all models* $\langle M, [\![]\!] \rangle$, $\langle M, [\![]\!] \rangle \models x_1, \ldots, x_n \Rightarrow y$

The S^{inf}-completeness of $\mathrm{L}^{(/,\backslash,\bullet)}$ actually entails RES-completeness, but the direct proof of RES completeness is however much easier. S^{inf}-completeness of $\mathrm{L}^{(/,\backslash)}$ entails RES-completeness. The adaptation of the direct proof from the $\mathrm{L}^{(/,\backslash,\bullet)}$ case fails.

The move from string semantics to residuated semi-groups involved one kind of abstraction, abandoning the idea that the value of a category is a set of strings. The *ternary frame semantics* represents another direction of abstraction, in which categories are still interpreted as sets of strings, but the interrelation of these items is abstractly defined by a 3-place 'accessibility' relation.

Definition 5 (Associative Ternary Frame) $\langle W, R \rangle$ *is an associate ternary frame if W is a non-empty set, R a ternary relation on W satisfying,*

$$\forall x, y, z, u \in W(\exists s(Rxys \wedge Rszu) \text{ iff } \exists t(Rxtu \wedge Ryzt))$$

Definition 6 (Associative Ternary Frame Model) $\langle W, R, [\![]\!] \rangle$ *is an associative ternary frame model if* $\langle W, R \rangle$ *is an associative ternary frame, and* $[\![]\!]$ *assigns the categories subsets of W, subject to:*

$$[\![a\bullet b]\!] = \{x : \exists y, z(Ryzx \wedge y \in [\![a]\!] \wedge z \in [\![b]\!])\}$$
$$[\![a\backslash b]\!] = \{x : \forall y, z(Ryxz \wedge y \in [\![a]\!] \longrightarrow z \in [\![b]\!])\}$$
$$[\![b/a]\!] = \{x : \forall y, z(Rxyz \wedge y \in [\![a]\!] \longrightarrow z \in [\![b]\!])\}$$

If we say a sequent is *satisfied by a model*, $\langle W, R, [\![]\!] \rangle \models x_1, \ldots, x_n \Rightarrow y$, iff $[\![x_1\bullet \ldots \bullet x_n]\!] \subseteq [\![y]\!]$ then

Theorem 4 (Dosen 1990). $\mathrm{L}^{(/,\backslash,\bullet)} \vdash a \Rightarrow b$ *iff every interpretation relative to every associative ternary frame satisfies* $a \Rightarrow b$

From a S^{inf} model a ternary frame model can quickly be constructed, taking the strings as the worlds and the accessibility relationship as $R(a, b, a\bullet b)$. Thus completeness of $\mathrm{L}^{(/,\backslash,\bullet)}$ and $\mathrm{L}^{(/,\backslash)}$ wrt the S^{inf} class implies completeness wrt ternary frames.

Analogs of the 3 above-mentioned model classes for polymorphic calculi will be considered in later sections, the primary emphasis being on the string semantic models.

1.4 Connection between grammars and string-semantic models

The question arises as to whether the model-theoretic results have any linguistic significance. Certainly amongst categorial grammarians the *string-semantics* results are treated as legitimising the sequent rules of $\mathrm{L}^{(/,\backslash)}$ and $\mathrm{L}^{(/,\backslash,\bullet)}$. Thus a linguistic problem which seems to not be Lambek solvable is typically not taken as a prompt to consider further, string-semantically *invalid* sequents, but as a prompt to *extend* the categorial language[4].

In this section I argue that a rationale can be given to this way of proceeding. Consider the sceptical position first. The sceptic says that the mere fact that a given set of sequents are exactly the M-valid sequents for *a* particular notion of model M provides no rationale for the use of these and only these sequents for defining grammars. For the semantics to provide such a rationale it must *reflect* the intended linguistic application. A parallel case is provided by Peano arithmetic. There is *a* notion of model according to which what is first order provable from the Peano axioms is exactly what is true in all models of the axioms. As an axiomatisation of arithmetic, however, one is still entitled to expect more than this to follow from the axioms, as the notion of model countenances *non-standard* models of the axioms.

We try to answer the sceptic now, considering the string-semantics. We will concentrate on $\mathrm{L}^{/,\backslash}$. Given an $\mathrm{L}^{/,\backslash}$-grammar $G \subseteq V^+ \times \mathcal{L}(/,\backslash)$ and an *interpretation* $[\![\cdot]\!] : \mathcal{L}(/,\backslash) \to 2^{V^+}$ one can ask the questions whether the *interpretation extends the grammar*:

$$(x)_G \subseteq [\![x]\!], \text{ for all } x \in \mathcal{L}(/,\backslash)$$

and the converse for whether the *grammar extends the interpretation.*

We consider first conditions under which it will obtain that an interpretation extends the grammar. An easy proof using the soundness of $\mathrm{L}^{/,\backslash}$ wrt string-semantic interpretations gives that if the model extends the values $(x)_{G_0}$, then it extend the values $(x)_G$.

Lemma 5. *If $G \subseteq V^+ \times \mathcal{L}(/,\backslash)$ is an $\mathrm{L}^{/,\backslash}$-grammar, and $[\![\cdot]\!] \in \mathcal{L} \to 2^{V^+}$ is the interpretation of a $\mathcal{L}(/,\backslash)$-model, then*

$$\textit{if } (x)_{G_0} \subseteq [\![x]\!] \textit{ for all } x \in \mathcal{L}(/,\backslash) \textit{ then } (x)_G \subseteq [\![x]\!] \textit{ for all } x \in \mathcal{L}(/,\backslash)$$

[4] The Combinatory Categorial Grammar school is an exception to this, preferring to keep the language fixed, and increase the admitted sequents

This makes the string-semantics relevant to the definition of G if we see G as attempting to give the fullest possible picture of what must hold in all interpretations which extend a given lexicon G_0.

We assume we are seeking to analyse a given language, say English, or any language for which there is no pre-existing *definition*. We assume that the categorisations facts about this language can be represented as the values of an interpretation $[\![\cdot]\!]$ on the *atomic categories* of $\mathcal{L}(/, \backslash)$. This is actually to assume nothing, as the valuation of *atomic* categories in a string-semantic model can be freely chosen .

Our to-be-analysed language has no pre-existing definition, and hence the interpretation of $\mathcal{L}(/, \backslash)$ is unknown. However certain facts will obtain which the analysis should respect. These may be individual categorisations (eg. 'a is an s') or generalisations (eg. 'whenever b is followed by an s an s results'). These then are constraints on the $\mathcal{L}(/, \backslash)$-interpretation that we are seeking to specify (eg. $a \in [\![\mathrm{s}]\!]$, $b \in [\![\mathrm{s/s}]\!]$). We would now like to know what further categorisations holds in all models which extend these lexical constraints. We can argue that the G we obtain from a lexicon (eg. $G_0 = \{(a, s), (b, s/s)\}$), by applying the definition of an $\mathrm{L}^{(/,\backslash)}$-grammar, gives us as good a picture as possible of this.

We have just seen that the *soundness* of $\mathrm{L}^{(/,\backslash)}$ gives that what G delivers is guaranteed to hold in all models which extend the lexicon. Thus G makes claims which can be relied upon concerning what holds in all models of the lexicon. With the soundness of $\mathrm{L}^{(/,\backslash)}$, if some $(a, x) \in G$ is a miscategorisation, we can assume that one of the lexical assumptions is mistaken. Suppose that $\mathrm{L}^{(/,\backslash)}$ were not *complete*, so that for some sequent representing an inclusion true in all models, $\mathrm{L}^{(/,\backslash)}$ did not derive it. Closing G_0 with this inclusion gives categorisations which hold in all models which extend G_0. Thus a G based on a complete calculus will give a fuller picture of what holds in all models of the lexicon than a G based on an incomplete calculus. With G based on an incomplete calculus, it will arise that some genuine consequence of a lexical hypothesis remains concealed.

Thus soundness of $\mathrm{L}^{(/,\backslash)}$ is desirable if G is to give reliable information about what holds in all models of the lexicon. Completeness of $\mathrm{L}^{(/,\backslash)}$ is desirable if G is not to miss information which obviously holds in all models of the lexicon.

We have that G is a member of the set $\{R \mid R \text{ is a subrelation of all interpretations extending } G_0\}$. The greatest element and therefore least upper bound of this set is clearly the intersection of interpretations extending G_0. A remaining open question[5] is whether G is also the least upper bound of this set:

Question 6. G is equal to the intersection of interpretations extending G_0.

An important aspect of the above discussion was that G was regarded as an *approximation* of the categorisation facts concerning the language to be analysed. Other considerations enter in if we would like to take G as 'the final answer'. In this case one takes the values of $(x)_G$ for atomic x to define $[\![x]\!]$. Given that G, including $(x)_G$ for atomic x, represents at best the *intersection* of all

[5] First posed by Hans Leiss, p.c.

interpretations extending G_0, there is no reason to assume that the interpretation based on $(x)_G$ for atomic x is also an interpretations extending G_0.

Buszkowski[4] gives the name *correctness* to the property of being a G such that when the interpretation is defined by $[\![x]\!] = (x)_G$ for atomic x, the interpretation extends the lexicon, and hence by Lemma 5, extends G. In the light for the foregoing discussion, this choice of term is an unfortunate one. A grammar G which is not in Buszkowski's sense *correct*, need not be seen as in any sense *wrong*, paradoxical as this may sound: the grammar completely and faithfully says what must hold in all models of the lexicon. When a grammar is not correct in Buszkowski's sense this simply means it is not model defining.

Our earlier example of a Lambek grammar *is* correct. The interpretation defined by $[\![x]\!] = (x)_G$ for atomic x, is the interpretation which was our example string-semantic model, and the model extends the lexicon. Not every Lambek grammar is correct. Consider an attempt to prove from $[\![x]\!] = (x)_G$, for x atomic, that the interpretation extends G, that is $(x)_G \subseteq [\![x]\!]$. We try to make an induction on the complexity of categories. Suppose for some α we have $\alpha \in (x/y)_G$. We require $\alpha \in [\![x/y]\!]$. Thus for all $\beta \in [\![y]\!]$, we require $\alpha\beta \in [\![x]\!]$. However, by induction we can hope only to have $(y)_G \subseteq [\![y]\!]$. For those $\beta \in (y)_G \cap [\![y]\!]$, we have $\alpha\beta \in (x)_G$, and hence by induction $\alpha\beta \in [\![x]\!]$, which is what we require. However, for those $\beta \in [\![y]\!] - (y)_G$, we have no argument for $\alpha\beta \in (x)_G$, and hence none for $\alpha\beta \in [\![x]\!]$.

For such negatively occurring y we need $[\![y]\!] \subseteq (y)_G$, in addition to $(y)_G \subseteq [\![y]\!]$. Let us define $arg(x/y) = arg(y\backslash x) = \{y\} \cup arg(x)$, with $arg(x) = \emptyset$, for atomic x, and $Lex(G_0) = \{x : \text{there is } \alpha \in V, \text{ with } (\alpha, x) \in G_0\}$

Theorem 7. *If* $[\![x]\!] = (x)_G$, *for atomic* x, *and* $(y)_G = [\![y]\!]$ *for all* $y \in \bigcup arg(x)_{x \,\in\, Lex(G_0)}$, *then* $[\![\cdot]\!]$ *extends G (i.e. is correct).*

Proof. (Sketch) Define $val(x/y) = \{x/y\} \cup val(x)$, with $val(x) = \{x\}$, for atomic x. One shows $(x)_G \subseteq [\![x]\!]$, for all $x \in \bigcup val(y)_{y \,\in\, Lex(G_0)}$, which suffices by Lemma 5 given $Lex(G_0) \subseteq \bigcup val(y)_{y \,\in\, Lex(G_0)}$. □

Buszkowski discusses two ways in which the condition in this theorem could get fulfilled. One is that all $x \in \bigcup arg(y)_{y \in Lex(G_0)}$ are atomic. The other is that for all subtypes of $x \in Lex(G_0)$, there is a lexical item occurring in $(x)_{G_0}$, and this lexical item occurs in no $(y)_{G_0}$ for $x \neq y$. The condition in the lemma is then fulfilled because it then holds that $(x)_G = [\![x]\!]$ for *all* subtypes of $x \in Lex(G_0)$. The following theorem is essentially Theorem 3 of [4], whose proof we omit.

Theorem 8 (Buszkowski,1982). *If* $[\![x]\!] = (x)_G$ *for atomic* x, *and for all subtypes of* $x \in Lex(G_0)$, *there is* $\alpha \in (x)_{G_0}$ *with* $\alpha \notin (y)_{G_0}$ *for* $x \neq y$, *then for all subtypes* y *of* $x \in Lex(G_0)$, $[\![y]\!] = (y)_G$.

It is to be noted that Theorem 8 shows $(x)_G = [\![x]\!]$ only for $x \in \bigcup sub(y)_{y \in Lex(G_0)}$. Buszkowski goes on to show that there it is never the case that $(x)_G = [\![x]\!]$, for *all* x. We noted earlier that our example of a Lambek model is obtained from the our example of a Lambek grammar by setting $[\![x]\!] = (x)_G$ for atomic x. We noted also that the model extends G. A counterexample to $(x)_G = [\![x]\!]$ is provided by np/(s/s), where $[\![\text{np}/(\text{s}/\text{s})]\!] = \{ab^p \mid\ p \geq 0\}$, and $(\text{np}/(\text{s}/\text{s}))_G = \emptyset$.

2 Polymorphic Grammars and Models

Polymorphism is an aspect of linguistic formalisms deserving study in its own right, appearing as it does either explicitly or implicitly in many of them. There is explicit appeal in 'cross-categorial' approaches to coordination, and implicit appeal in the lexical entries of unification based frameworks: quantifying over all consistent extensions of a given feature structure, or over all specialisations of a first order term. In previous work, I have investigated various aspects of a polymorphic extension of the original Lambek grammar formalism. So, in addition to basic categories, one has *category variables*, and where x is a category, so also is $QX.x$, $Q \in \{\forall, \exists\}$. The rules for quantifiers are given in Figure 1

See [16] and [12] for proofs that derivability is preserved under substitution throughout for a free variable, and under change of a bound variable. A basic linguistic motivation for the extended calculus is that Chomsky's (now proven) conjecture that the Lambek calculus characterises only CF languages, is not true of the polymorphic extension [7]. Also concretely various linguistic phenomena including quantification, coordination, and extraction are tackled via polymorphism in [6], [9], [10]. Via a decidability result for a particular class of sequents, [8] also shows how these polymorphic accounts may be handled computationally, though [11] proves undecidability in the general case. Our main aim here is to study whether model theoretic results for the Lambek calculus generalise to its basic polymorphic extension. Using again Definition 1 of L-grammar, we give the following example of a polymorphic grammar

Example 4: Polymorphic grammar : let $V = \{a, b, c\}$, $\mathcal{L} = \{/, \backslash, \forall\}$, $\mathrm{L} = \mathrm{L}^{(/,\backslash,\forall)}$, and let G_0 be given by the following table:

$$G_3 \begin{cases} a & t/(t/e) & t/(e\backslash t) & \forall X.t/(X\backslash t)/(X/e) \\ b & t/(t/(e\backslash s)) & t/((e\backslash s)\backslash t) & \forall X.t/(X\backslash t)/(X/(e\backslash s)) \\ c & t/(t/(s\backslash t)) & t/((s\backslash t)\backslash t) & \forall X.t/(X\backslash t)/(X/(s\backslash t)) \\ & t/(t/(s\backslash(t\backslash t))) & t/((s\backslash(t\backslash t))\backslash t) & \forall X.t/(X\backslash t)/(X/(s\backslash(t\backslash t))) \end{cases}$$

then $(t)_G = \{perm(abc)^p \mid p \geq 1\}$ (a non-CF language, see [7])

2.1 Polymorphic String Semantic Models

In the monomorphic case, models were defined to include an interpretation function $[\![]\!]$ defined on all categories, whose operation on complex categories was, however, completely determined by its operation on atomic categories. In the polymorphic case it is more convenient to isolate as I a partial function defined only on basic categories.

Definition 7 (Polymorphic string-semantic model) *$\langle S, \bullet, \mathcal{V}, I\rangle$ is a* polymorphic string-semantic *model if $\langle S, \bullet\rangle$ is a free semigroup, $\mathcal{V} \subseteq \mathcal{P}(S)$, $\mathcal{V}$ non-empty, and I maps basic categories into $\mathcal{V}$.*

$\mathcal{V}$ will be referred to as the *range of quantification*. To handle variables in the polymorphic case, the denotation of a category is defined with respect to a model (see above) and an *assignment*, where this a function in $\mathcal{V}^{VAR}$. If g is an assignment, and A a set in $\mathcal{V}$, then g^A_X is the unique assignment h such that (i) $h(Y) = g(Y)$, for $Y \neq X$, and (ii) $h(X) = A$ otherwise.

Definition 8 (Denotation) *If $M = \langle S, \bullet, \mathcal{V}, I\rangle$ is polymorphic string-semantic model, and g an assignment in $\mathcal{V}^{VAR}$, then $[\![x]\!]^g$ is defined as follows*
1. for basic x, $[\![x]\!]^g = I(x)$
2. for variables X, $[\![X]\!]^g = g(X)$
3. $[\![x/y]\!]^g = \{a \in S : \forall b \in [\![y]\!]^g, a\bullet b \in [\![x]\!]^g\}$
4. $[\![y\backslash x]\!]^g = \{a \in S : \forall b \in [\![y]\!]^g, b\bullet a \in [\![x]\!]^g\}$
5. $[\![\forall X.y]\!]^g = \bigcap\{[\![y]\!]^{g^A_X} : A \in \mathcal{V}\}$
6. $[\![\exists X.y]\!]^g = \bigcup\{[\![y]\!]^{g^A_X} : A \in \mathcal{V}\}$

Clearly for categories lacking free variables, the denotation does not depend on the assignment. Satisfaction is now defined with respect to a model and an assignment: $M, g \models x_1, \ldots, x_n \Rightarrow y$ iff $[\![x_1]\!]^g\bullet \ldots \bullet[\![x_n]\!]^g \subseteq [\![y]\!]^g$.

The question with which we will be concerned is *which* subsets of S occur in $\mathcal{V}$. Three conditions that one might impose on the range of quantification $\mathcal{V}$ of a polymorphic string-semantic model $\langle S, \bullet, \mathcal{V}, I\rangle$ are:

Condition 1 $\mathcal{V} = \mathcal{P}(S)$
Condition 2 $\mathcal{V}$ *covers the categories*:
$\{[\![x]\!]^g : x$ a category, g an assignment$\} \subseteq \mathcal{V}$
Condition 3 $\mathcal{V}$ is *covered by the closed categories*:
$\mathcal{V} \subseteq \{[\![x]\!]^g : x$ is a closed category, g is an assignment$\}$

PS^{inf}_1, PS^{inf}_2 and $PS^{inf}_{2,3}$ indicate that particular conditions on $\mathcal{V}$ are in force. Note Condition 1 entails Condition 2. The simplest option, that *all* subsets of S should be available turns out to be incorrect; it is for this reason a range of quantification is given as an explicit parameter in the definition of the model [6]. We also note for PS^{inf}_2 models, that I does not affect whether we have a PS^{inf}_2 model: it suffices if we have $[\![x]\!]^g \in \mathcal{V}$, for all assignments and *constant-free* categories, as the denotation of a constant-containing category will coincide with that of some constant-free category.

2.1.1 PS^{inf}_1-Incompleteness of $L^{(/,\backslash,\forall,\exists)}$

The PS^{inf}_1 class chooses the quantifier range to be simply the power set of the underlying set of strings, the most obvious candidate for the semantics of

[6] Thus our models are very akin to the so-called *general models* of 2nd Order logic, introduced by [15].

the quantified calculi. We show now PS_1^{inf}-incompleteness of $\mathrm{L}^{(/,\backslash,\forall,\exists)}$. Not all instances of the schemata in the lefthand column of the table below are derivable (if for example x and y are assumed to be atomic then one can easily see that all attempts at a proof will fail). Yet all instances of these schemata are satisfied in all PS_1^{inf} models, for the reasons indicated, in the righthand column (where M is an arbitrary PS_1^{inf} model $\langle S, \bullet, I\rangle$, and g an arbitrary assignment):

Satisfied in all M, g	*Holds for any M, any g*
1. $\forall X.X/X \Rightarrow y$	$[\![\forall X.X/X]\!]^g = \emptyset$
2. $\forall X.X\backslash X/X \Rightarrow y$	$[\![\forall X.X\backslash X/X]\!]^g = \emptyset$
3. $\forall X.(X/y)\backslash X \Rightarrow y$	$[\![\forall X.(X/y)\backslash X]\!]^g \subseteq [\![y]\!]^g$
4. $x \Rightarrow \exists X.y/(X/X)$	$[\![\exists X.y/(X/X)]\!]^g = S$
5. $x \Rightarrow \exists X.y/(X\backslash X/X)$	$[\![\exists X.y/(X\backslash X/X)]\!]^g = S$

Proof. Let M be an arbitrary PS_1^{inf} model $\langle S, \bullet, I\rangle$, and g an arbitrary assignment

Equations 1, 4 : We note that $\{b\}/\{b\} = \emptyset$, where $b \in S$. For suppose $a \in \{b\}/\{b\}$. Then $a\bullet b \in \{b\}$, which implies $a\bullet b = b$, contradicting that the algebras are free.[7] So, when $B = \{b\}$, $[\![X/X]\!]^{g^B_X} = \emptyset = [\![\forall X.X/X]\!]^g$. Since for any $A \subseteq S$, we have $A/\emptyset = S$, we have $[\![y/(X/X)]\!]^{g^B_X} = S = [\![\exists X.y/(X/X)]\!]^g = S$.

Equations 2, 5 : $\{b\}\backslash\{b\}/\{b\} = \emptyset$, since supposing $a \in \{b\}\backslash\{b\}/\{b\}$, gives $b\bullet a\bullet b = b$, contradicting that the algebras are free. So where $B = \{b\}$, $[\![X\backslash X/X]\!]^{g^B_X} = \emptyset = [\![\forall X.X\backslash X/X]\!]^g$. Also $[\![y/(X\backslash X/X))]\!]^{g^B_X} = S = [\![\exists X.y/(X\backslash X/X)]\!]^g$.

Inclusion 3 : First suppose $[\![y]\!]^g$ is empty. For all B we require $[\![\forall X.((X/y)\backslash X)]\!]^g \subseteq [\![(X/y)\backslash X]\!]^{g^B_X}$, but when $B = \emptyset$, the right-hand side is $S\backslash\emptyset$, which is empty. Hence $[\![\forall X.((X/y)\backslash X)]\!]^g$ is empty. Now assume $[\![y]\!]^g$ is non-empty, and suppose $q \in [\![\forall X.(X/y)\backslash X]\!]^g$, and $q \notin [\![y]\!]^g$. For all $B \subseteq S$, we have $q \in [\![(X/y)\backslash X]\!]^{g^B_X}$. Choose $B = b\bullet[\![y]\!]$, for some b. Then $b \in [\![X/y]\!]^{g^B_X}$, and so we require $b\bullet q \in b\bullet[\![y]\!]^g$, which implies that $q \in [\![y]\!]^g$.

□

These counterexamples can change when empty antecedents are permitted and an identity element is included, but counterexamples remain. $\forall X.((X\backslash X)/X)$ is empty when $S \neq \{\epsilon\}$ and otherwise denotes $\{\epsilon\}$. The same is true of $\forall X.((X\backslash X\backslash X)/X)$, and hence $\forall X.((X\backslash X)/X) \Rightarrow \forall X.((X\backslash X\backslash X)/X)$ is valid. It is not derivable.

We do not wish to extend the calculus to suit the PS_1^{inf} semantics. We would like to give the category $\forall X.(X\backslash X)/X$ to `and`, but a complete calculus must allow any y to be derived from this category, so that `and` would satisfy the subcategorisation requirements of any word. Under the assumption that $\mathcal{V} = \mathcal{P}(S)$, `and`:$\forall X.(X\backslash X)/X$ means says every set of strings is closed under

[7] More generally when there is a longest string in B, $B/B = \emptyset$

the operation of inserting and between members, which is not simply not true. Restricting to sets of strings that are the denotation of a *category*, however, closure under insertion of and is more reasonable. So for sets *which are the value of a category* we suggest that $\mathcal{V}$ just contain the values of categories, which Conditions 2 and 3 give different ways of formalising.

2.1.2 PS_2^{inf} Soundness and Completeness of $\mathrm{L}^{(/,\backslash,\forall)}$

In this section we prove:

Theorem 9. $\mathrm{L}^{(/,\backslash,\forall)} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all models* $M \in PS_2^{inf}$, *and assignments* g, $M, g \models x_1, \ldots, x_n \Rightarrow y$

We note that for the soundness direction, $\bullet$ and $\exists$ can easily be included. For the completeness direction, the inclusion of $\bullet$ brings problems well known from the monomorphic case. The inclusion of $\exists$ also brings problems, discussed later in the paper.

Proof. PS_2^{inf}-soundness of $\mathrm{L}^{(/,\backslash,\forall)}$ is proved by induction on the size of a proof. In the following (M, g) is always a pair with $M = \langle S, \bullet, \mathcal{V}, I \rangle \in PS_2^{inf}$, and $g \in \mathcal{V}^{VAR}$.

Consider a proof of $T \Rightarrow x$. The leaves are instances of $x \Rightarrow x$, and are therefore satisfied by every (M, g). It suffices to show that the rules of the calculus preserve this property of being satisfied by every (M, g). The cases for Cut, /L,/R,\L,\R are minor alterations of the corresponding cases in the proof of the string-semantic soundness of $\mathrm{L}^{(/,\backslash)}$, and we give just the quantifier cases, including $\exists$. The argument for $\forall$L and $\exists$R uses that $\mathcal{V}$ meets Condition 2.

Case: $\forall$L. Suppose we obtain the sequent U, $\forall X.x$, $V \Rightarrow w$ from U, $x[y/X]$, $V \Rightarrow w$, that the premise is satisfied by every (M, g) and that the conclusion is not. Hence for some (M, g), there are $\mathbf{u} \in [\![U]\!]^g$, $a \in [\![\forall X.x]\!]^g$, $\mathbf{v} \in [\![V]\!]^g$ such that $\mathbf{u}a\mathbf{v} \notin [\![w]\!]^g$. Since $a \in [\![\forall X.x]\!]^g$, then for any $B \in \mathcal{V}$, $a \in [\![x]\!]^{g^X_B}$. Because $M \in PS_2^{inf}$, we have $[\![y]\!]^g \in \mathcal{V}$, and choosing $B = [\![y]\!]^g$, we have $[\![x]\!]^{g^X_B} = [\![x[y/X]]\!]^g$. Therefore $a \in [\![x[y/X]]\!]^g$, and therefore U, $x[y/X]$, $V \Rightarrow w$ is not satisfied by (M, g), which is a contradiction.
Case: $\exists$R (similar to $\forall$L). Suppose we obtain the sequent $T \Rightarrow \exists X.x$ from $T \Rightarrow x[y/X]$, that the premise is satisfied by every (M, g) and that the conclusion is not. Hence for some (M, g), there are $\mathbf{t} \in [\![T]\!]^g$, such that $\mathbf{t} \notin [\![\exists X.x]\!]^g$. So for no $B \in \mathcal{V}$, $\mathbf{t} \in [\![x]\!]^{g^X_B}$. Because $M \in PS_2^{inf}$, we have $[\![y]\!]^g \in \mathcal{V}$, and choosing $B = [\![y]\!]^g$, we have $[\![x]\!]^{g^X_B} = [\![x[y/X]]\!]^g$. Therefore $\mathbf{t} \notin [\![x[y/X]]\!]^g$, and therefore $T \Rightarrow x[y/X]$ is not satisfied by (M, g), which is a contradiction.
Case:$\forall$R. Suppose we obtain the sequent $T \Rightarrow \forall X.x[X/Z]$, where $Z \notin FV(T)$, and $X \notin FV(\forall Z.x)$, from $T \Rightarrow x$, and that the premise is satisfied by every (M, g) and that the conclusion is not. Hence for some (M, g), there is $\mathbf{t} \in [\![T]\!]^g$, $\mathbf{t} \notin [\![\forall X.x[X/Z]]\!]^g$. Therefore for some $B \in \mathcal{V}$, $\mathbf{t} \notin [\![x[X/Z]]\!]^{g^X_B}$. By the choice

of X, $[\![x[X/Z]]\!]^{g^X_B} = [\![x]\!]^{g^Z_B}$. Also since $Z \notin FV(T)$, $[\![T]\!]^g = [\![T]\!]^{g^Z_B}$. Therefore we have $\mathbf{t} \in [\![T]\!]^{g^Z_B}$, and $\mathbf{t} \notin [\![x]\!]^{g^Z_B}$. So we have that $T \Rightarrow x$ is not a satisfied by (M, g^Z_B), which is a contradiction.
Case:$\exists$L (similar to $\forall$R). Suppose we obtain the sequent $U, \exists X.x[X/Z], V \Rightarrow w$, where $Z \notin FV(U,V,w)$, and $X \notin FV(\exists Z.x)$, from $U,\ x,\ V \Rightarrow w$, and that the premise is satisfied by every (M,g) and that the conclusion is not. Hence for some (M,g) there are $\mathbf{u} \in [\![U]\!]^g$, $\mathbf{v} \in [\![V]\!]^g$, $a \in [\![\exists X.x[X/Z]]\!]^g$, such that $\mathbf{u}a\mathbf{v} \notin [\![w]\!]^g$. For some $B \in \mathcal{V}$, $a \in [\![x[X/Z]]\!]^{g^X_B}$. By the choice of X, $[\![x[X/Z]]\!]^{g^X_B} = [\![x]\!]^{g^Z_B}$. Also since $Z \notin FV(U,V,w)$, $[\![U]\!]^g = [\![U]\!]^{g^Z_B}$, $[\![V]\!]^g = [\![V]\!]^{g^Z_B}$, $[\![w]\!]^g = [\![w]\!]^{g^Z_B}$. Therefore we have $\mathbf{u} \in [\![U]\!]^{g^Z_B}$, $a \in [\![x[y/X]]\!]^{g^Z_B}$, $\mathbf{v} \in [\![V]\!]^{g^Z_B}$, and $\mathbf{u}a\mathbf{v} \notin [\![w]\!]^{g^Z_B}$, so $U, x, V \Rightarrow w$ is not satisfied by (M, g^Z_B), which is a contradiction.

We now prove PS_2^{inf}-completeness of $\mathrm{L}^{(/,\backslash,\forall)}$. S^{inf} completeness of $\mathrm{L}^{(/,\backslash)}$can be proved, by taking the semigroup of sequences of categories under the operation of sequence concatenation. The set of sequences which derive a category x, which we notate as $\mathcal{A}[x]$ ('antecedents of x'), is then used to give a canonical interpretation. The following proof pursues the same strategy, with an added twist owing to the presence of *assignments*. We define a canonical model, $M^c = \langle S^c, \bullet^c, \mathcal{V}^c, I^c \rangle$ thus:

S^c = all non-empty sequences of categories, $\bullet^c$ = sequence concatenation
$\mathcal{V}^c = \{\mathcal{A}[x] : x \text{ is a category}\}$
$I^c(x) = \mathcal{A}[x]$, where x is a basic category

Note that $\mathcal{P}(S^c) \neq \mathcal{V}^c$: no finite set can be $\mathcal{A}[x]$ for example. Thus $M^c \notin PS_1^{inf}$, which is good, otherwise the properties of M^c would contradict our claim of PS_1^{inf}-incompleteness.

We relate every assignment g from variables to values in $\mathcal{V}^c$ to (an equivalence class) of *substitutions*. We say a substitution σ 'IS ALLOWED BY g' if for each variable, X, $g(X) = \mathcal{A}[\sigma(X)]$. σ refers here to an *infinite simultaneous substitution*, mapping each variable to some category. Such a σ is extended in the obvious way to a function applying to every category[8]. Note that by the definition of 'IS ALLOWED BY', more than one substitution may be allowed by g. We first observe the following concerning $\mathcal{A}[x]$:

Lemma 10. *If* $\mathcal{A}[x] = \mathcal{A}[x']$, *then* $\mathcal{A}[x/y] = \mathcal{A}[x'/y]$, $\mathcal{A}[y/x] = \mathcal{A}[y/x']$, $\mathcal{A}[x\bullet y] = \mathcal{A}[x'\bullet y]$, $\mathcal{A}[y\bullet x] = \mathcal{A}[y\bullet x']$, $\mathcal{A}[QX.x] = \mathcal{A}[QX.x']$.

Proof If $\mathcal{A}[x] = \mathcal{A}[x']$, then $x \Rightarrow x'$ and vice-versa. Using these one can show $x/y \Rightarrow x'/y$, $y/x \Rightarrow y/x'$, $x\bullet y \Rightarrow x'\bullet y$, $y\bullet x \Rightarrow y\bullet x'$, $QX.x \Rightarrow QX.x'$, and vice-versa. From these, the desired identities of antecedents follow. □

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[8] In application to a quantified term, $\forall Y.y$, such that Y occurs free in $\sigma(X_i)$ for one of $X_i \in FV(\forall Y.y)$, there is a change of bound variable to the first variable not in $FV(\forall Y.y)$ nor $\sigma(X_i)$ for any $X_i \in FV(\forall Y.y)$.

We define a relation F whose first argument is a pair $\langle x, g\rangle$ consisting of a category x and an assignment g from variables to values in $\mathcal{V}^c$, and whose second argument is a member of $\mathcal{V}$:

$$F(\langle x, g\rangle, A) \text{ iff } A = \mathcal{A}[\sigma(x)]\ , \text{ for some } \sigma \text{ allowed by } g$$

F is a function We show that if σ_1, σ_2 are two substitutions allowed by the same assignment (that is for all variables, $\mathcal{A}[\sigma_1(X_i)] = \mathcal{A}[\sigma_2(X_i)]$), then $\mathcal{A}[\sigma_1(x)] = \mathcal{A}[\sigma_2(x)]$. The substitutions σ_1 and σ_2 may precipitate changes of bound variable in x, but because the antecedents of categories which differ from each other only by a change of bound variable are identical, we can assume that σ_1 and σ_2 are such as to cause no change of bound variable. Then we use Lemma 10 for each of the free-variables of x, and for each of its occurrences, in order to infer that $\mathcal{A}[\sigma_1(x)] = \mathcal{A}[\sigma_2(x)]$. Henceforth we write $F(x, g)$ for the unique A such that $F(\langle x, g\rangle, A)$.

$F(x, g) = [\![x]\!]^g$ We show now the denotation function $[\![]\!]$ and F are the same function.

Case: variable, X.
$F(X, g)$
$= \mathcal{A}[\sigma(X)]$ for any σ allowed by g
$= \mathcal{A}[\sigma(X)]$ where $g(X) = \mathcal{A}[\sigma(X)]$
$= g(X)$

Case: basic category, x. $F(x, g)$
$= \mathcal{A}[\sigma(x)]$, for any σ allowed by g
$= \mathcal{A}[x]$, because $x = \sigma(x)$
$= I(x) = [\![x]\!]^g$

Case: $[\![x/y]\!]^g$. Let σ be an arbitrary substitution allowed by g, and write $\tilde{x}$ for $\sigma(x)$, and $\tilde{y}$ for $\sigma(y)$. We need to show that
$\{T \in S : \forall T_1 \in S(\text{if } L \vdash T_1 \Rightarrow \tilde{y}, \text{then } L \vdash T, T_1 \Rightarrow \tilde{x})\} = \{T \in S : L \vdash T \Rightarrow \tilde{x}/\tilde{y}\}$.
Left to Right. Note, $L \vdash \tilde{y} \Rightarrow \tilde{y}$, hence $L \vdash T,\ \tilde{y}\ \Rightarrow \tilde{x}$, hence $L \vdash T \Rightarrow \tilde{x}/\tilde{y}$ (by /R)
Right to Left. Let T_1 be arbitrarily chosen such that $L \vdash T_1 \Rightarrow \tilde{y}$. Hence $L \vdash \tilde{x}/\tilde{y}, T_1 \Rightarrow \tilde{x}$ (by /L), hence $L \vdash T, T_1 \Rightarrow \tilde{x}$ (by Cut, assuming $L \vdash T \Rightarrow \tilde{x}/\tilde{y}$)

Case $[\![\forall X.x]\!]^g$. Let σ be an arbitrary substitution allowed by g. Let $\sim$ denote the relation between categories when they differ by changes of bound variable. Because $\mathcal{A}[z] = \mathcal{A}[z']$, when $z \sim z'$, we can assume that $\forall X.x$ is such that the substitution σ causes no changes of bound variable. Let σ^z_X be the substitution differing from σ only by assigning z to X. We need:
$\cap\{\mathcal{A}[\sigma^y_X(x)] : \mathcal{A}[y] \in \mathcal{V}\} = \mathcal{A}[\sigma(\forall X.x)]$.
Left to Right. Suppose for all $\mathcal{A}[y] \in \mathcal{V}$, $T \in \mathcal{A}[\sigma^y_X(x)]$. Pick $Z \notin FV(T, \sigma(Y))$, where $Y \in FV(\forall X.x)$. We have $L \vdash T \Rightarrow \sigma^Z_X(x)$, hence $L \vdash T \Rightarrow \forall Z.\sigma^Z_X(x)$. By the choice of Z, $\forall Z.\sigma^Z_X(x) \sim \sigma(\forall X.x)$. Hence $L \vdash T \Rightarrow \sigma(\forall X.x)$.
Right to Left. Suppose $T \in \mathcal{A}[\sigma(\forall X.x]$, i.e. $L \vdash T \Rightarrow \forall X.\sigma^X_X(x)$. Hence for all y, $L \vdash T \Rightarrow (\sigma^X_X(x))[y/X]$ (By Cut, and $\forall$L). Suppose $T \notin \cap\{\mathcal{A}[\sigma^y_X(x)] : \mathcal{A}[y] \in \mathcal{V}\}$. Then for some $\mathcal{A}[y] \in \mathcal{V}$, $T \notin \mathcal{A}[\sigma^y_X(x)]$. But $\sigma^y_X(x) \sim (\sigma^X_X(x))[y/X]$, therefore $L \not\vdash T \Rightarrow (\sigma^X_X(x))[y/X]$ which is a contradiction.

M^c is in PS_2^{inf} Since $[\![x]\!]^g = F(x, g) = \mathcal{A}[\sigma(x)]$, for some σ allowed by g, we

have that $[\![x]\!]^g \in \mathcal{V}^c$, and therefore $M^c \in PS_2^{inf}$.

<u>*M^c is a countermodel to underivable sequents*</u> Suppose for all g, $[\![x_1]\!]^g \bullet \ldots \bullet [\![x_n]\!]^g \subseteq [\![y]\!]^g$. Let σ^* be an identity substitution. Let g^* be such that $g^*(X) = \mathcal{A}[X]$, then σ^* is allowed by g^*. We have $L \vdash x_i \Rightarrow x_i$. Because $\mathcal{A}[x_i] = \mathcal{A}[\sigma^*(x_i)] = F(x_i, g^*)$ $= [\![\sigma^*(x_i)]\!]^{g^*} = [\![x_i]\!]^{g^*}$, we have $x_i \in [\![x_i]\!]^{g^*}$. Therefore under the supposition, we have $x_1, \ldots, x_n \in [\![y]\!]^{g^*} = \mathcal{A}[\sigma^*(y)] = \mathcal{A}[y]$, and therefore $L \vdash x_1, \ldots, x_n \Rightarrow y$. $\square$

2.1.3 Examples of $PS_{2,3}^{inf}$ Models

Completeness for the $PS_{2,3}^{inf}$ class is an open question. We make the modest contribution of showing that models in this class exist [9].

Example Six: *S is any set of strings closed under concatenation. $\mathcal{V} = \{S\}$, the interpretation is such that for basic categories, $I(x) = S$.* For Condition 2, we simply check that $\mathcal{V}$ is closed under $/,\backslash$, and $\cap$ and $\cup$. Because $\mathcal{V}$ contains only one set, Condition 2 entails Condition 3.

Example Seven:*S is any set of strings closed under concatenation, $\mathcal{V} = \{S, \emptyset\}$, the interpretation is such that for basic categories $I(x) \in \mathcal{V}$.* We note $[\![\exists X.X]\!]^g = S \cup \emptyset = S$, and that $[\![\forall X.X]\!]^g = S \cap \emptyset = \emptyset$. Thus $\mathcal{V}$ meets Condition 3. For Condition 2, we check that $\mathcal{V}$ is closed under $/,\backslash$, and pairwise $\cap$, and $\cup$. This gives that unions and intersections over arbitrary subsets of $\mathcal{V}$ are contained in $\mathcal{V}$.

Example Eight: *S: $\{\alpha, \beta\}^+$, $\mathcal{V} = \{\{\alpha\beta\}, \{\alpha\}, \{\beta\}, \emptyset, S\}$, and we assume that for some basic category,* s, *$I(\mathrm{s}) = \{\alpha\beta\}$, for some basic category,* np, *$I(\mathrm{np}) = \{\alpha\}$, and that all further basic categories have a value identical to one of these.*

For Condition 3 we note that $\mathcal{V} = \{[\![\mathrm{s}]\!]^g, [\![\mathrm{np}]\!]^g, [\![\mathrm{np}\backslash\mathrm{s}]\!]^g, [\![\forall X.X]\!]^g, [\![\exists X.X]\!]^g\}$. Note $\mathcal{V}$ is not closed under the union of *arbitrary* subsets. For Condition 2, however, we require less than this. Define a *spectrum* of x, relative to an assignment g and a variable X as $\{[\![x]\!]^{g^X_A} : A \in \mathcal{V}\}$. Condition 2 may be reformulated as the requirement that all spectra are subsets of $\mathcal{V}$, and this we will show, considering only categories with no vacuous quantification (vacuous quantifiers can be discarded without changing the denotation). The property is entailed by:

all spectra sp_1, sp_2 of x are (i) in $\mathcal{P}(\mathcal{V})$ and (ii) if x is complex then for all $B_1, B_2 \in \{\{\alpha\}, \{\beta\}, \{\alpha\beta\}\}$, $B_1 \in sp_1$, $B_2 \in sp_2$ implies $B_1 = B_2$. (1)

Proof. We show this by induction on the complexity of a category x. Consider variables and basic categories. Clearly the spectra of these are subsets of $\mathcal{V}$, and

[9] The examples furnish also further PS_2^{inf} models, in addition to the example M^c of the completeness proof.

so we have (i). Because x is not complex, (ii) is trivially true. Now for induction assume we have the property for all categories of complexity less than n and consider a category, x of complexity n. We give only the $\exists X.y$ case.

Case $x = \exists X.y$. Let sp be a spectrum of $\exists X.y$, for some assignment g and variable Z. Consider $[\![\exists X.y]\!]^{g_Z^A}$, for some $A \in \mathcal{V}$. This is $\cup sp_A$, where sp_A is the spectrum of y for g_Z^A and the variable X. By induction, $sp_A \subseteq \mathcal{V}$. When y is not complex, then sp_A is either a singleton subset of $\mathcal{V}$ or $\mathcal{V}$ itself, and so $\cup sp_A \in \mathcal{V}$. When y is complex, then by induction we have that sp_A contains at most one member of $\{\{\alpha\},\{\beta\},\{\alpha\beta\}\}$. This gives that $\cup sp_A \in \mathcal{V}$. Therefore $sp \subseteq \mathcal{V}$, i.e. (i). We must now show (ii) because $\exists X.y$ is complex. First note that if y is not complex, $y = X$, and the unique spectrum of $\exists X.y$ is $\{S\}$, and so (ii) is trivially satisfied. So suppose y is complex and suppose $B_1, B_2 \in \{\{\alpha\},\{\beta\},\{\alpha\beta\}\}$, such that $B_1 \in sp_1(\exists X.y)$ and $B_2 \in sp_2(\exists X.y)$ and suppose that $B_1 \neq B_2$. This implies spectra sp'_1 and sp'_2 of y and $B'_1, B'_2 \in \{\{\alpha\},\{\beta\},\{\alpha\beta\}\}$, such that $B'_1 \neq B'_2$, which contradicts our inductive assumption.

□

2.1.4 Open questions

Existential Quantifier The above completeness proof does not extend to the case with $\exists$. As with $\bullet$, the canonical model does not conform to the condition imposed by the connective. Consider $\exists X.a/(X/X)$, and assume that it contains no free variables. In the canonical model we need that $[\![\exists X.a/(X/X)]\!]^g = \mathcal{A}[\sigma(\exists X.a/(X/X))]$, for all σ allowed by g. Since $\exists X.a/(X/X) \in \mathcal{A}[\exists X.a/(X/X)]$, we require $\exists X.a/(X/X) \in [\![\exists X.a/(X/X)]\!]^g$, which holds iff $\exists X.a/(X/X) \in [\![a/(X/X)]\!]^{g_X^A}$ for some $A = \mathcal{A}[y] \in \mathcal{V}$, which holds iff $\exists X.a/(X/X) \Rightarrow a/(y/y)$ for some y. But there is no such y. For supposing there was such a y, then consideration of possible proofs gives that for some variable Z not occurring in y, we have $L \vdash Z \Rightarrow y$ and $L \vdash y \Rightarrow Z$. By the soundness of L, then these two sequents are satisfied in every model and assignment, and this could only be the case if every model had a singleton range of quantification, which is not the case.

Finite set of atomic strings M^c is not in PS_2. For the monomorphic calculus, there is a construction assigning to each underivable sequent a countermodel, taking V to be the subcategories of the given sequent. For V categories, the set of antecedents *over* V is the value, and the interpretation is extended to non-V categories. For $\mathrm{L}^{(/,\backslash,\forall)}$ isolating the interpretation of V categories from non-V is impossible: the quantified V categories depend on $\mathcal{V}$, which itself must contain the values of all categories. PS_2 completeness is therefore an open question.

2.2 Connection between polymorphic grammars and string-semantics

In a similar way to the monomorphic case, we can establish a connection between the model-theory and the grammars. Since we have defined grammars with respect to a finite vocabulary, the most relevant model class is PS_2. It is reasonable to assume that in G_0, lexical items are assigned closed categories, and that the interesting part of the $\mathrm{L}^{/,\backslash,\forall}$-closure G concerns also closed categories.

Given an $\mathrm{L}^{/,\backslash,\forall}$-grammar G and a polymorphic model $\langle S, \bullet, \mathcal{V}, I\rangle$, we say the model *extends the grammar* when:

$$(x)_G \subseteq [\![x]\!]^g, \text{ for all (closed) } x \in \mathcal{L}(/, \backslash, \forall)$$

and the converse when the grammar extends the model. As in the monomorphic case, we have that if the model extends the values of $(x)_{G_0}$, then it extends the values $(x)_G$.

PS_2 soundness follows from PS_2^{inf}, and this gives that what G delivers is guaranteed to hold in all PS_2 models which extend the lexicon.

Suppose that $\mathrm{L}^{(/,\backslash,\forall)}$ were not PS_2^{inf}-complete, so that for some sequent representing an inclusion true in all models, $\mathrm{L}^{(/,\backslash,\forall)}$ did not derive it. Closing G_0 with this inclusion gives categorisations which are present in all PS_2^{inf}-models. Thus a G based on a complete calculus will give a fuller picture of what holds in all models of the lexicon than a G based on an incomplete calculus.

Whilst PS_2-completeness remains an open question, we still do not know whether G gives an incomplete picture of what holds in all PS_2 models.

The polymorphic pendant of Buszkowski's notion of a correct grammar has not yet been explored by the present author. In the polymorphic case, there are two parameters to be extracted from the grammar, the interpretation and the range of quantification. For $\mathcal{V}$ there are two possibilities: either $\mathcal{V} = \{(x)_G \mid x \text{ is closed category}\}$, or $\mathcal{V} \subset \{(x)_G \mid x \text{ is a closed category}\}$. The latter possibility is motivated by the fact that in a typical polymorphic grammar, the grammar will define a $(x)_G$, for many complex x which play no role in the derivations of the categorisations that one is primarily interested in. A model may most easily be obtained in such cases fulfilling $(x)_G \subseteq [\![x]\!]^g = S$.

2.3 Other Models for Quantified Calculi

We considered a polymorphic analog of the residuated semigroup models. For the quantified calculi we assign a value to a category relative to an assignment as usual, and interpret universal and existential quantifiers via greatest lower and least upper bounds.

Definition 9 (*PRES* model) *$\langle M, \bullet, /, \backslash, [\![]\!]\rangle$ is a RES-model if: $\langle M, \bullet, /, \backslash\rangle \in$ RES, and $[\![]\!]$ assigns members of M to categories in accordance to:*
1. $[\![C(x,y)]\!]^g = \mathbf{C}([\![x]\!]^g, [\![y]\!]^g)$.
2. $[\![x]\!]^g = g(x)$, if x is variable
3. $[\![x]\!]^{g_1} = [\![x]\!]^{g_2}$, if x is basic
4. $[\![\forall X.x]\!]^g = g.l.b\ (\{[\![x]\!]^{g^A_X} : A \in M\})$
5. $[\![\exists X.x]\!]^g = l.u.b\ (\{[\![x]\!]^{g^A_X} : A \in M\})$

Theorem 11. $\mathrm{L}^{(/,\backslash,\bullet,\forall,\exists)} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all models $\langle M, \bullet, /, \backslash, [\![]\!]\rangle \in$ PRES, and assignments g, $[\![x_1]\!]^g, \ldots, [\![x_n]\!]^g \leq [\![y]\!]^g$*

Proof. A canonical model $\langle M, /, \backslash, \bullet, [\![]\!]\rangle$ is defined with: $M = \{\mathcal{A}[x] : x$ is a category$\}$, $\mathbf{C}(\mathcal{A}[x], \mathcal{A}[y]) = \mathcal{A}[C(x,y)]$, for $\mathbf{C} = /, \backslash, \bullet$, $\mathcal{A}[x] \leq \mathcal{A}[y]$ iff $x \Rightarrow y$, and $[\![x]\!]^g = \mathcal{A}[\sigma(x)]$, for any substitution σ allowed by g. We show that this is a PRES-model.

Case x is basic: the interpretation is clearly independent of assignment.
Case x is a variable, X. Suppose $g(X) = \mathcal{A}[y]$. $[\![X]\!]^g = \mathcal{A}[\sigma(X)] = \mathcal{A}[y]$
Case Where C is a binary connective $[\![C(x,y)]\!]^g$. Let σ be allowed by g. We require $[\![C(x,y)]\!]^g = \mathbf{C}([\![x]\!]^g, [\![y]\!]^g)$. $\mathbf{C}([\![x]\!]^g, [\![y]\!]^g) = \mathbf{C}(\mathcal{A}[\sigma(x)], \mathcal{A}[\sigma(y)])=$ $\mathcal{A}[C(\sigma(x), \sigma(y))] = \mathcal{A}[\sigma(C(x,y))]$.
Case $[\![\forall X.x]\!]^g$. We require that $\mathcal{A}[\sigma(\forall X.x)]$ is the g.l.b. of $\{\mathcal{A}[\sigma^y_X(x)] : \mathcal{A}[y] \in M\}$.
$\mathcal{A}[\sigma(\forall X.x)]$ is a l.b.: we require that for any y, $\mathcal{A}[\sigma(\forall X.x)] \leq \mathcal{A}[\sigma^y_X(x)]$, i.e $L \vdash \sigma(\forall X.x) \Rightarrow \sigma^y_X(x)$. We can assume without loss of generality that σ precipitates no changes of bound variables in $\forall X.x$, i.e. $\sigma(\forall X.x) = \forall X.\sigma^X_X(x)$. Therefore by a ($\forall$L) inference, $L \vdash \sigma(\forall X.x) \Rightarrow \sigma^y_X(x)$.
$\mathcal{A}[\sigma(\forall X.x)]$ is a g.l.b.: Let $\mathcal{A}[z]$ be a l.b. of $\{\mathcal{A}[\sigma^y_X(x)] : \mathcal{A}[y] \in M\}$. That is suppose for any y, $L \vdash z \Rightarrow \sigma^y_X(x)$. If Z is some variable not free in x, nor $\sigma(Y)$, where $Y \in FV(x)$, then $\sigma(\forall X.x) \sim \forall Z.\sigma^Z_Z(x[Z/X]) \sim \forall Z.\sigma^Z_X(x)$. Let Z be also not free in z. We have $L \vdash z \Rightarrow \sigma^Z_X(x)$, and by the choice of Z, we have $L \vdash z \Rightarrow \forall Z.\sigma^Z_X(x)$. Therefore $L \vdash z \Rightarrow \sigma(\forall X.x)$, and $\sigma(\forall X.x)$ is the g.l.b.
Case $[\![\exists X.x]\!]^g$: similar to $\forall X.x$

It is easy to show that for any underivable sequent there is an assignment that leaves it unsatisfied: the assignment allows a null substitution. This completes the proof of Theorem 11. $\square$

It is to be noted that Theorem 11 concerns the calculus with product, whereas Theorem 9 concerns a product-free calculus. The $PRES$-completeness of the *product-free* calculi remains an open question.

We consider also a polymorphic version of the associative ternary frame semantics.

Definition 10 (Polymorphic associative ternary frame interpretation) *$\langle W, R, \mathcal{V}, [\![]\!]\rangle$ is quantified associative ternary frame model if conditions on W and R from Definition 6 obtain, $\mathcal{V} \subseteq \mathcal{P}(W)$, $[\![]\!]$ is subject to the further conditions;*

$[\![x]\!]^g = g(x)$ *of x is a variable*
$[\![x]\!]^g$ *is independent of g if x is basic*
$[\![\forall X.x]\!]^g = \cap\{[\![x]\!]^{g^A_X} : A \in \mathcal{V}\}$
$[\![\exists X.x]\!]^g = \cup\{[\![x]\!]^{g^A_X} : A \in \mathcal{V}\}$

Theorem 12. $\mathrm{L}^{(/,\backslash,\bullet,\forall,\exists)} \vdash x_1, \ldots, x_n \Rightarrow y$ *iff for all polymorphic associative ternary frame interpretations, where $\mathcal{V}$ covers the categories, $[\![x_1 \bullet \ldots \bullet x_n]\!]^g \subseteq [\![y]\!]^g$*

The proof is omitted here. Soundness is easily shown by induction on the size of derivations. Completeness is easily shown by combining the well known canonical model construction for associative ternary frames with the technique used in the proof of Theorem 9.

3 Directions for future work

Okada [17] proves completeness results for an adaptation of Girard's Phase Space semantics [13], [14]. In the terminology of [17], the monomorphic (quantifier-free) case involves an *Intuitionistic Phase Space*, $D \subseteq 2^S$, S being a commutative monoid, and with D closed under linear implication ($= /$), and arbitrary intersections. S^{inf} models for $\mathrm{L}^{(/,\backslash)}$ can probably be seen as a version of this lacking commutativity, a unit, closure under intersection, and with closure under $/,\backslash$. A *classical* phase spaces require $A = \perp/(\perp/A)$, for all A in D, for some distinguished member $\perp$ of D. For the monomorphic case, soundness and completeness results for phase spaces have been obtained for linear logic in both classical [13] and intuitionistic [1] [17] variants.

In contrast to the string-semantics, $x \otimes y$ is treated as the smallest element of D containing $x \times y$ (the same as $\perp/(\perp/(x \times y))$ in classical phase spaces). Pentus' tricky proof [18] of S-completeness of $\mathrm{L}^{(/,\backslash,\bullet)}$ can probably be seen as completeness for a non-commutative variant of intuitionistic phase-spaces, with D additionally closed under $\times$.

Okada proves soundness and completeness of second order linear logic for *2nd Order Phase Spaces*, where one additionally has (for the numbering and notation refer to [17]):

P4 : every formula A is associated with a subset $\langle A\rangle$ of D, known as the candidate

P5 : for every formula B, for every $\alpha \in \langle B\rangle$, $A^*[\alpha/X] \in \langle A[B/X]\rangle$

and universal quantification is then handled via:

L10 : for any $\zeta : D \to D$, $\forall X.\zeta(X) = \bigcap\{\zeta(\alpha) \mid \alpha \in \langle B\rangle,\ B \text{ a formula}\}$

The exact connection between Okada's results and the results here is a topic for future work, and we end with some speculations concerning this. Perhaps our $\mathcal{V}$ can be seen as the union of the candidates of a second order phase space. PS_2^{inf} models and second order phase spaces share the feature that quantification is not handled by quantification over arbitary subsets of the underlying algebra. Can our PS_1^{inf}-incompleteness result be seen as an incompleteness result for second order phase spaces, where $D = 2^S$? Besides dropping commutativity, the PS_2^{inf} models also lack any requirement that $\mathcal{V}$ be closed under arbitary intersections. Requiring that quantifier ranges are closed under $/,\backslash$, and $\cap$, gives our Condition 2, so PS_2^{inf}-soundness gives soundness for the PS^{inf} models with such a closure. Whether completeness also holds for such models, and whether this is essentially Okada's (intuitionistic) result remain open questions.

Complete or not under such closures, there remains a question whether such an intersection is linguistically plausible. Requiring such a closure can be expected to make it harder to find a PS model extending a given polymorphic grammar: the universal quantifiers range over more sets, and correspondingly the denotations of the quantified categories get smaller.

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