Probability Basics

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Outline

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standard frequentist interpretation is that the systems can be observed over and over again, and that the relative frequency of \( X = a \) in all the observations tends to a stable fixed value as the number of observations tends to infinity. \( P(X = a) \) is this limit

\[
P(X = a) = \lim_{N \to \infty} \frac{freq(X = a)}{N}
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the relative freq. of (2 or 4 or 6) is by definition the same as the

$$(\text{rel.freq. 2}) + (\text{rel.freq. 4}) + (\text{rel.freq. 6})$$. So its not surprising that by definition the probability of an 'event' is the sum of the mutually exclusive atomic possibilities that are contained within it (ie. ways for it to happen) so

$$P(X = 2 \lor X = 4 \lor X = 6) = P(X = 2) + P(X = 4) + P(X = 6)$$
Independence of two events

- suppose two 'events' $A$ and $B$. If the probability of $A \land B$ occurring is just the probability $A$ occurring times the probability of $B$ occurring, you say the events $A$ and $B$ are independent

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- This is defined to be

<table>
<thead>
<tr>
<th>Conditional Prob</th>
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**Product Rule**

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$$P(A \land B) = P(A|B)P(B)$$

- since $P(A|B)P(B) = P(B|A)P(A)$, you also get the famous

**Bayesian Inversion**

$$P(A|B) = \frac{P(A \land B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$
Alternative expressions of independence

- recall independence was defined to be $P(A \land B) = P(A) \times P(B)$. Given the definition of conditional probability there are equivalent formulations of independence in terms of conditional probability:
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NOTE: each of these on its own is equivalent to $P(A \land B) = P(A) \times P(B)$
Suppose > 1 feature/attribute of your system/situation eg. rolling a red & a green dice. Using $X$ for red & $Y$ for green can specify events with their values and their probs with expressions such as:\(^1\)

$$P(X = 1, Y = 2)$$

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if $A$ is range of values for $X$ & $B$ is range for $Y$, the must have

$$\sum_{a \in A, b \in B} P(X = a, Y = b) = 1$$

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can wish to consider the probs of events specified by the value on just one feature (eg. those where $X=1$) and the probs. of these are called marginal probabilities and are obtained by summing the joints with all possible values of the other feature

$$P(X = 1) = \sum_{b \in B} P(X = 1, Y = b)$$

\[1\] note comma often used instead of $\wedge$
the conditional probability function for two features $X$ and $Y$ is

$$P(X|Y) = \frac{P(X, Y)}{P(Y)}$$

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you say $P(X|Y) = P(X)$ and the features $X$ and $Y$ are independent in case for every value $a$ for $X$ and $b$ for $Y$ you have

$$\frac{P(X = a, Y = b)}{P(Y = b)} = P(X = a)$$
Chain Rule

- generalising to more variables, you can derive the indispensable chain rule

\[
P(X, Y, Z) = P(Z|(X, Y)) \times P(X, Y) = P(Z|(X, Y)) \times P(Y|X) \times P(X)
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Notation: typically \( P(Z|(X, Y)) \) is written as \( P(Z|X, Y) \)
Conditional Independence

- there is a notion of **conditional independence**. It may be that two variables $X$ and $Y$ are not in general independent, but given a value for a third variable $Z$, $X$ and $Y$ become independent.

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- Real-life cases of this arise where $Z$ describes a *cause*, which manifests itself into two *effects* $X$ and $Y$, which though very dependent on $Z$, do not directly influence each other.

- The theories behind Speech Recognition and Machine Translation typically make a lot of *conditional independence* assumptions.