4CSLL5
IBM Translation Models

Martin Emms

October 4, 2018
Parameter learning (brute force)

Introduction
The brute force EM algorithm defined
A formula for $p(a|o, s)$
Examples brute force EM in action
Brute force EM learning
Outline

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Learning Lexical Translation Models
Learning Lexical Translation Models

We would like to estimate the lexical translation probabilities \( t(o|s) \) from a parallel corpus \( (o^1, s^1) \ldots (o^D, s^D) \)
Learning Lexical Translation Models

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▶ this would be easy if we had the alignments ie. $(o^1, a^1, s^1) \ldots (o^D, a^D, s^D)$ (or just how frequent \ldots)
Learning Lexical Translation Models

- We would like to estimate the lexical translation probabilities $t(o|s)$ from a parallel corpus $(o_1, s_1) \ldots (o_D, s_D)$
- this would be easy **if we had the alignments** ie. $(o_1, a_1, s_1) \ldots (o_D, a_D, s_D)$ (or just how frequent \ldots )
- but we don’t ...
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- but we don’t . . .

- if we *knew the parameters*, it would be (relatively) easy to calculate the 'odds' on alignments ie. $P(a^1|o^1, s^1) \ldots P(a^D|o^D, s^D)$
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- but we don’t ... 
- something of a 'Chicken and Egg' situation
- but the EM algorithm embraces this exactly
EM Algorithm roughly

Expectation Maximization (EM) in a nutshell

1. initialize model parameters (e.g. uniform)
2. assign probabilities to the missing data
3. treat probabilities like counts in complete data and estimate model parameters from the pseudo-completed data
4. iterate steps 2–3 until convergence
The EM algorithm keeps re-estimating the parameters. The following slides show in a graphical fashion the evolution of the parameters when the process is applied to the corpus

<table>
<thead>
<tr>
<th>$s^1$</th>
<th>la maison</th>
<th>$s^2$</th>
<th>la maison bleu</th>
<th>$s^3$</th>
<th>la fleur</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o^1$</td>
<td>the house</td>
<td>$o^2$</td>
<td>the blue house</td>
<td>$o^3$</td>
<td>the flower</td>
</tr>
</tbody>
</table>

and with all $tr(o|s)$ values initially equal
Parameter learning (brute force)

Introduction

initial

la ma the ho

la ma the blu ho

la fle the flo
after one
after two
after four
after ten
Outline

Parameter learning (brute force)

Introduction

The brute force EM algorithm defined

A formula for $p(a|o,s)$

Examples brute force EM in action
Parameter learning (brute force)

The brute force EM algorithm defined
to arrive at the EM algorithm for this case its a good idea to first spell out explicitly what the counting and parameter-estimation would look like if you *had* the alignments
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then migrate that into the EM version replacing anything which assume a definite alignment with lines which consider all possible alignments, treating each has having a 'count' of $p(a|o, s)$
to arrive at the EM algorithm for this case it's a good idea to first spell out explicitly what the counting and parameter-estimation would look like if you *had* the alignments

then *migrate* that into the EM version replacing anything which assume a definite alignment with lines which consider all possible alignments, treating each has having a 'count' of $p(a|o,s)$

next 2 slides do exactly this
Estimating translation probs $tr(o|s)$ from complete data

Suppose you have a corpus of $D$ pairs of sentence, and each has an alignment $a$. From this we can estimate the values of $tr(o|s)$ for the model in a straightforward way\(^1\)

\[ COUNT \]

---

\(^1\)If we wanted to be really thorough we could set up the differential equations which define the parameters which will maximise the likelihood of the data under the model and show that solving them for $tr(o|s)$ parameters amounts to the counting procedure shown.
Estimating translation probs $tr(o|s)$ from complete data

Suppose you have a corpus of $D$ pairs of sentence, and each has an alignment $a$. From this we can estimate the values of $tr(o|s)$ for the model in a straightforward way$^1$

\[
\text{COUNT}
\]

for each $o \in V_o$

for each $s \in V_s \cup \{\text{NULL}\}$

set $\#(o, s) = 0$

$^1$If we wanted to be really thorough we could set up the differential equations which define the parameters which will maximise the likelihood of the data under the model and show that solving them for $tr(o|s)$ parameters amounts to the counting procedure shown
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**COUNT**

for each $o \in V_o$
    for each $s \in V_s \cup \{NULL\}$
        set $\#(o,s) = 0$

for each aligned pair $(o,a,s)$ // just counting freqs of $(o,s)$
    for each $j \in 1: \ell_o$ // word-pairs in the data
        $\#(o_j,s_{a(j)}) += 1$

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for each $j \in 1: l_o$ // word-pairs in the data

$\#(o_j, s_{a(j)}) += 1$

**TAKE RATIOS**

---

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**TAKE RATIOS**
for each $s \in \mathcal{V}_s \cup \{\text{NULL}\}$
    for each $o \in \mathcal{V}_o$
        $tr(o|s) = \frac{(o,s)}{\sum_o (o,s)}$

$^1$If we wanted to be really thorough we could set up the differential equations which define the parameters which will maximise the likelihood of the data under the model and show that solving them for $tr(o|s)$ parameters amounts to the counting procedure shown
outline of brute-force EM training for IBM model 1

initialise $tr(o|s)$ uniformly

repeat $[E]$ followed by $[M]$ till convergence
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outline of brute-force EM training for IBM model 1

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[E]
for each $o \in V_o$
  for each $s \in V_s \cup \{NULL\}$
    set $\#(o, s) = 0$

for each pair $(o, s)$
  for each $a$ calculate $p(a|o, s)$  // pseudo counts of $(o,s)$ word pairs
    for each $j \in 1: \ell_o$
      $\#(o_j, s_{a(j)}) += p(a|o, s)$  // in virtual data
outline of brute-force EM training for IBM model 1

initialise $tr(o|s)$ uniformly

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for each pair \( (o, s) \)
  for each \( a \) calculate \( p(a|o, s) \) // pseudo counts of \((o, s)\) word pairs
    for each \( j \in 1 : \ell_o \) // in virtual data
      \( #(o_j, s_{a(j)}) += p(a|o, s) \)

[M]
for each \( s \in \mathcal{V}_s \cup \{\text{NULL}\} \)
  for each \( o \in \mathcal{V}_o \)
    \( tr(o|s) = \frac{ #(o, s) }{ \sum_{o} #(o, s) } \)
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Brute force EM for IBM Model 1 contd: formula for $p(a|o, s)$

to implement this need to be able to calculate $p(a|o, s)$ for each possible alignment $a$ between $o$ and $s$. 
Brute force EM for IBM Model 1 contd: formula for $p(a|o, s)$

To implement this, we need to be able to calculate $p(a|o, s)$ for each possible alignment $a$ between $o$ and $s$.

By definition this is

$$
\frac{p(o, a, s)}{\sum_{a'} p(o, a', s)} \tag{8}
$$
Brute force EM for IBM Model 1 contd: formula for $p(a|o, s)$

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We have a formula for the combinations of $\langle o, a, s \rangle$, ie.

$$
P(o, a, \ell_o, s) = p(s) \times \frac{p(\ell_o|\ell_s)}{(\ell_s + 1)^{\ell_o}} \times \prod_j [p(o_j|s_{a(j)})]
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$$\frac{p(o, a, s)}{\sum_{a'} p(o, a', s)}$$  \hspace{1cm} (8)

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and when plugged into the numerator and denominator in (8), the $p(s)$ and $\frac{p(\ell_o|\ell_s)}{(\ell_s + 1)^{\ell_o}}$ terms cancel,
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$$
p(a|o, s) = \frac{\prod_j [p(o_j|s_{a(j)})]}{\sum_{a'} \prod_j [p(o_j|s_{a'(j)})]}
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Brute force EM for IBM Model 1 contd: formula for $p(a|o, s)$

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$$p(a|o, s) = \frac{\prod_j [p(o_j|s_{a(j)})]}{\sum_{a'} \prod_j [p(o_j|s_{a'(j)})]}$$  \hspace{1cm} (9)

so we can deploy (9) for $p(a|o, s)$ in the brute-force EM algorithm, and thereby iteratively (re)-estimate the translation probabilities.
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Examples brute force EM in action
A brute force example

see Labs/brute_force_ibm_model1_worked_eg.pdf for detailed worked through of this assuming a corpus of 2 pairs

<table>
<thead>
<tr>
<th>$s^1$</th>
<th>la maison</th>
<th>$s^2$</th>
<th>la fleur</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o^1$</td>
<td>the house</td>
<td>$o^2$</td>
<td>the flower</td>
</tr>
</tbody>
</table>

initialising all $tr(o|s)$ uniformly to $\frac{1}{3}$

note: to keep calcs. to manageable size makes slight simplification of not allowing any alignments from o to a NULL added to s: this does not affect the validity of the formula (9)
## Evolution of the translation probabilities $tr(o|s)$

### $o|s$ at each iteration

<table>
<thead>
<tr>
<th>Obs</th>
<th>Src</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>final</th>
</tr>
</thead>
<tbody>
<tr>
<td>the</td>
<td>la</td>
<td>0.33</td>
<td>0.5</td>
<td>0.6</td>
<td>0.69</td>
<td>0.77</td>
<td>0.84</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>house</td>
<td>la</td>
<td>0.33</td>
<td>0.25</td>
<td>0.2</td>
<td>0.15</td>
<td>0.11</td>
<td>0.081</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>flower</td>
<td>la</td>
<td>0.33</td>
<td>0.25</td>
<td>0.2</td>
<td>0.15</td>
<td>0.11</td>
<td>0.081</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>the</td>
<td>maison</td>
<td>0.33</td>
<td>0.5</td>
<td>0.43</td>
<td>0.36</td>
<td>0.3</td>
<td>0.24</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>house</td>
<td>maison</td>
<td>0.33</td>
<td>0.5</td>
<td>0.57</td>
<td>0.64</td>
<td>0.7</td>
<td>0.76</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>flower</td>
<td>maison</td>
<td>0.33</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>the</td>
<td>fleur</td>
<td>0.33</td>
<td>0.5</td>
<td>0.43</td>
<td>0.36</td>
<td>0.3</td>
<td>0.24</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>house</td>
<td>fleur</td>
<td>0.33</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>flower</td>
<td>fleur</td>
<td>0.33</td>
<td>0.5</td>
<td>0.57</td>
<td>0.64</td>
<td>0.7</td>
<td>0.76</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
Evolution of corpus-related statistics

- EM is guaranteed to increase the data probability – the probability with hidden variables summed out, which in full is

\[
\prod_d [p(o^d, s^d)] = \prod_d \left[ \sum_a \left[ p(o^d, a|\ell_o^d, s) \right] \times p(\ell_o^d|\ell_s^d) \times p(s^d) \right]
\]
Evolution of corpus-related statistics

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\]

- the length probability \( p(\ell_o^d|\ell_s^d) \) and the source probability \( p(s^d) \) are not being updated in the algorithm, so its sufficient to track the product of the \( \sum_a \left[ p(o^d, a|\ell_o^d, s) \right] \) terms, which is

\[
\prod_d \left[ \sum_a \frac{1}{(\ell_s^d + 1)^{\ell_o^d}} \times \prod_j \text{tr}(o_j|s_{a(j)}) \right]
\] (10)
Evolution of corpus-related statistics

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\]

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\[
\prod_d \left[ \sum_a \frac{1}{(\ell_s^d + 1)^{\ell_o^d}} \times \prod_j tr(o_j|s_{a(j)}) \right] \tag{10}
\]

- This quantity should monotonically increase over iterations.
Evolution of corpus-related statistics

- EM is guaranteed to increase the data probability — the probability with hidden variables summed out, which in full is

$$\prod_d \left[ p(o^d, s^d) \right] = \prod_d \left[ \sum_a \left[ p(o^d, a|\ell^d_0, s) \right] \times p(\ell^d_0|\ell^d_s) \times p(s^d) \right]$$

- the length probability $p(\ell^d_0|\ell^d_s)$ and the source probability $p(s^d)$ are not being updated in the algorithm, so its sufficient to track the product of the $\sum_a \left[ p(o^d, a|\ell^d_0, s) \right]$ terms, which is

$$\prod_d \left[ \sum_a \frac{1}{(\ell^d_s + 1)\ell^d_0} \times \prod_j tr(o_j|s_{a(j)}) \right] \quad \text{(10)}$$

- This quantity should monotonically increase over iterations.

- Practically speaking, the quantity in (10) though increasing will be minutely small, so that some alternatives are often used. If $p$ is just the probability, alternatives often used are $\log(p)$ — 'the log prob', $1/p$ — the 'perplexity', and $\log(1/p)$ — 'the log perplexity'
Evolution of corpus-related statistics (cont'd)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>final</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(o^d</td>
<td>s^d) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p(\text{the house}</td>
<td>la maison) )</td>
<td>0.11</td>
<td>0.19</td>
<td>0.2</td>
<td>0.21</td>
<td>0.22</td>
<td>...</td>
</tr>
<tr>
<td>( p(\text{the flower}</td>
<td>la fleur) )</td>
<td>0.11</td>
<td>0.19</td>
<td>0.2</td>
<td>0.21</td>
<td>0.22</td>
<td>...</td>
</tr>
</tbody>
</table>

Corpus level stats at each iteration:

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>prob</td>
<td>0.012</td>
<td>0.035</td>
<td>0.039</td>
<td>0.044</td>
<td>0.048</td>
<td>...</td>
<td>0.0625</td>
</tr>
<tr>
<td>log prob</td>
<td>-6.3</td>
<td>-4.8</td>
<td>-4.7</td>
<td>-4.5</td>
<td>-4.4</td>
<td>...</td>
<td>-4</td>
</tr>
<tr>
<td>perp</td>
<td>81</td>
<td>28</td>
<td>25</td>
<td>23</td>
<td>21</td>
<td>...</td>
<td>16</td>
</tr>
<tr>
<td>log perp</td>
<td>6.3</td>
<td>4.8</td>
<td>4.7</td>
<td>4.5</td>
<td>4.4</td>
<td>...</td>
<td>4</td>
</tr>
</tbody>
</table>

- The values shown for \( p(o|s) \) are really values for \( p(o|s,\ell_o) \). If \( \epsilon \) were the value of \( p(\ell_o|\ell_s) \), then the true values of \( p(o|s) \) would be these multiplied by \( \epsilon \).
- The values in the 'prob' row **increase**, as do the values in the 'log prob' row – they are always negative because the probabilities are always \(< 1\).
- Correspondingly, the values in the 'perp' row **always fall**, as they are just the inverses of the probabilities. The values in the 'log perp' row also fall.