Some worked eg's of Parameter Estimation with HMMs: a fully observed case and the 'brute-force' EM unsupervised case

Martin Emms

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Some worked eg's of Parameter Estimation with HMMs: a fully observed case and the 'brute-force' EM unsupervised case

Outline
In the following there is a worked example of an iteration of brute-force EM for the HMM case.
It all follows the general pattern of EM seen earlier in the hidden coin scenario, and the hidden alignment scenario.
Nonetheless it may help to see some of the specifics of what precisely has to be 'counted' in the HMM case
A fully observed (or ’supervised’) case of estimation

first suppose a fully observed corpus of obs and state sequences (states in blue, obs in red):

\[
\begin{align*}
\text{one: } & a \quad \text{three: } x \quad \text{four: } z \quad \text{five: } \# && (= o^0, s^0) \\
\text{one: } & a \quad \text{three: } y \quad \text{three: } x \quad \text{five: } \# && (= o^1, s^2) \\
\text{two: } & b \quad \text{four: } x \quad \text{one: } a \quad \text{three: } y \quad \text{five: } \# && (= o^2, s^3)
\end{align*}
\]
A fully observed (or 'supervised') case of estimation

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\begin{align*}
\text{one:} & \quad a \quad \text{three:} x \quad \text{four:} z \quad \text{five:} \# \quad (= o^0, s^0) \\
\text{one:} & \quad a \quad \text{three:} y \quad \text{three:} x \quad \text{five:} \# \quad (= o^1, s^2) \\
\text{two:} & \quad b \quad \text{four:} x \quad \text{one:} a \quad \text{three:} y \quad \text{five:} \# \quad (= o^2, s^3)
\end{align*}
\]

The values for \( \pi, A \) and \( B \) derived by relative frequencies in this corpus are:
A fully observed (or 'supervised') case of estimation

first suppose a fully observed corpus of obs and state sequences (states in blue, obs in red):

\[
\begin{align*}
\text{one}: & \text{a} \quad \text{three}: & \text{x} \\
\text{four}: & \text{z} \\
\text{five}: & \# \\
\text{one}: & \text{a} \\
\text{three}: & \text{y} \\
\text{five}: & \#
\end{align*}
\]

(\(= o^0, s^0\))

\[
\begin{align*}
\text{one}: & \text{a} \\
\text{three}: & \text{x} \\
\text{five}: & \#
\end{align*}
\]

(\(= o^1, s^2\))

\[
\begin{align*}
\text{two}: & \text{b} \\
\text{four}: & \text{x} \\
\text{one}: & \text{a} \\
\text{three}: & \text{y} \\
\text{five}: & \#
\end{align*}
\]

(\(= o^2, s^3\))

The values for \(\pi\), \(A\) and \(B\) derived by relative frequencies in this corpus are:

\[
\begin{align*}
A & \quad C(\text{one, three}) = 3 & C(\text{one}) = 3 & P(\text{three}|\text{one}) = 1 \\
& \quad C(\text{three, four}) = 1 & C(\text{three}) = 4 & P(\text{four}|\text{three}) = 0.25 \\
& \quad C(\text{three, three}) = 1 & P(\text{three}|\text{three}) = 0.25 \\
& \quad C(\text{three, five}) = 2 & P(\text{five}|\text{three}) = 0.5 \\
& \quad C(\text{two, four}) = 1 & C(\text{two}) = 1 & P(\text{four}|\text{two}) = 1 \\
& \quad C(\text{four, five}) = 1 & C(\text{four}) = 2 & P(\text{five}|\text{four}) = 0.5 \\
& \quad C(\text{four, one}) = 1 & P(\text{one}|\text{four}) = 0.5
\end{align*}
\]
A fully observed (or 'supervised') case of estimation

first suppose a fully observed corpus of obs and state sequences (states in blue, obs in red):

\[
\begin{align*}
\text{one: } a & \quad \text{three: } x & \quad \text{four: } z & \quad \text{five: } 
\quad \# & \quad (= o^0, s^0) \\
\text{one: } a & \quad \text{three: } y & \quad \text{three: } x & \quad \text{five: } 
\quad 
\# & \quad (= o^1, s^2) \\
\text{two: } b & \quad \text{four: } x & \quad \text{one: } a & \quad \text{three: } y & \quad \text{five: } 
\quad 
\# & \quad (= o^2, s^3)
\end{align*}
\]

The values for \( \pi \), \( A \) and \( B \) derived by relative frequencies in this corpus are:

\[
\begin{align*}
A & \quad C(\text{one, three}) = 3 & \quad C(\text{one}) = 3 & \quad P(\text{three}|\text{one}) = 1 \\
& \quad C(\text{three, four}) = 1 & \quad C(\text{three}) = 4 & \quad P(\text{four}|\text{three}) = 0.25 \\
& \quad C(\text{three, three}) = 1 & \quad & \quad P(\text{three}|\text{three}) = 0.25 \\
& \quad C(\text{three, five}) = 2 & \quad & \quad P(\text{five}|\text{three}) = 0.5 \\
& \quad C(\text{two, four}) = 1 & \quad C(\text{two}) = 1 & \quad P(\text{four}|\text{two}) = 1 \\
& \quad C(\text{four, five}) = 1 & \quad C(\text{four}) = 2 & \quad P(\text{five}|\text{four}) = 0.5 \\
& \quad C(\text{four, one}) = 1 & \quad & \quad P(\text{one}|\text{four}) = 0.5 \\
B & \quad C(\text{one, a}) = 3 & \quad C(\text{one}) = 3 & \quad P(a|\text{one}) = 1 \\
& \quad C(\text{three, x}) = 2 & \quad C(\text{three}) = 4 & \quad P(x|\text{three}) = 0.5 \\
& \quad C(\text{three, y}) = 2 & \quad & \quad P(y|\text{three}) = 0.5 \\
& \quad C(\text{two, b}) = 1 & \quad C(\text{two}) = 1 & \quad P(b|\text{two}) = 1 \\
& \quad C(\text{five, #}) = 3 & \quad C(\text{five}) = 3 & \quad P(#|\text{five}) = 1 \\
& \quad C(\text{four, z}) = 1 & \quad C(\text{four}) = 2 & \quad P(z|\text{four}) = 0.5 \\
& \quad C(\text{four, x}) = 1 & \quad & \quad P(x|\text{four}) = 0.5
\end{align*}
\]
A fully observed (or 'supervised') case of estimation

first suppose a fully observed corpus of obs and state sequences (states in blue, obs in red):

\[
\begin{align*}
one \colon & a \\
three \colon & x \\
four \colon & z \\
five \colon & \# \\
\text{one} : a & \text{three} : x \text{ four} : z \text{ five} : \# \quad (= o^0, s^0) \\
\text{one} : a & \text{three} : y \text{ three} : x \text{ five} : \# \quad (= o^1, s^2) \\
two : b & \text{four} : x \text{ one} : a \text{ three} : y \text{ five} : \# \quad (= o^2, s^3)
\end{align*}
\]

The values for \(\pi\), A and B derived by relative frequencies in this corpus are:

\[
\begin{align*}
A & \quad C(\text{one, three}) = 3 \quad C(\text{one}) = 3 \quad P(\text{three}|\text{one}) = 1 \\
& \quad C(\text{three, four}) = 1 \quad C(\text{three}) = 4 \quad P(\text{four}|\text{three}) = 0.25 \\
& \quad C(\text{three, three}) = 1 \quad P(\text{three}|\text{three}) = 0.25 \\
& \quad C(\text{three, five}) = 2 \quad P(\text{five}|\text{three}) = 0.5 \\
& \quad C(\text{two, four}) = 1 \quad C(\text{two}) = 1 \quad P(\text{four}|\text{two}) = 1 \\
& \quad C(\text{four, five}) = 1 \quad C(\text{four}) = 2 \quad P(\text{five}|\text{four}) = 0.5 \\
& \quad C(\text{four, one}) = 1 \quad P(\text{one}|\text{four}) = 0.5 \\
B & \quad C(\text{one, a}) = 3 \quad C(\text{one}) = 3 \quad P(\text{a}|\text{one}) = 1 \\
& \quad C(\text{three, x}) = 2 \quad C(\text{three}) = 4 \quad P(\text{x}|\text{three}) = 0.5 \\
& \quad C(\text{three, y}) = 2 \quad P(\text{y}|\text{three}) = 0.5 \\
& \quad C(\text{two, b}) = 1 \quad C(\text{two}) = 1 \quad P(\text{b}|\text{two}) = 1 \\
& \quad C(\text{five, #}) = 3 \quad C(\text{five}) = 3 \quad P(\text{#}|\text{five}) = 1 \\
& \quad C(\text{four, z}) = 1 \quad C(\text{four}) = 2 \quad P(\text{z}|\text{four}) = 0.5 \\
& \quad C(\text{four, x}) = 1 \quad P(\text{x}|\text{four}) = 0.5 \\
\pi & \quad C(s_1 = \text{one}) = 2 \quad P(s_1 = \text{one}) = 0.666 \\
& \quad C(s_1 = \text{two}) = 1 \quad P(s_1 = \text{two}) = 0.333
\end{align*}
\]
(brute-force) EM case (unsupervised)

Note that this estimation from a small set of examples has left quite a few options set to have zero probability. This will be handy in the following illustration of the unsupervised case as it keeps the number of alternatives down. In general such zeroes would be a bad idea. Let’s use these probs as initial values in some unsupervised estimation, just using a corpus of observed words

Let’s suppose this corpus to have much more of the final obs sequence than before:

1× axz#
1× ayx#
10× bxay#
brute-force EM: considering all the possible paths

On our initial settings what state sequences are possible for these observation sequences?

1 the 'hidden' states are not really very hidden!
brute-force EM: considering all the possible paths

On our initial settings what state sequences are possible for these observation sequences?

<table>
<thead>
<tr>
<th>observations</th>
<th>possible completions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( o^0 )</td>
<td>a × z #</td>
</tr>
<tr>
<td></td>
<td>one:a three:x four:z five:#</td>
</tr>
<tr>
<td></td>
<td>( (= o^0, s^0) )</td>
</tr>
<tr>
<td>( o^1 )</td>
<td>a y x #</td>
</tr>
<tr>
<td></td>
<td>one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>( (= o^1, s^1) )</td>
</tr>
<tr>
<td></td>
<td>one:a three:y three:x five:#</td>
</tr>
<tr>
<td></td>
<td>( (= o^1, s^2) )</td>
</tr>
<tr>
<td>( o^2 )</td>
<td>b x a y #</td>
</tr>
<tr>
<td></td>
<td>two:b four:x one:a three:y five:#</td>
</tr>
<tr>
<td></td>
<td>( (= o^2, s^3) )</td>
</tr>
<tr>
<td></td>
<td>ditto for ( o^3 \ldots o^{11} )</td>
</tr>
</tbody>
</table>

\(^1\)the 'hidden' states are not really very hidden!
brute-force EM: considering all the possible paths

On our initial settings what state sequences are possible for these observation sequences?

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<tbody>
<tr>
<td>$o^0$ a x z #</td>
<td>one:a three:x four:z five:# (= $o^0, s^0$)</td>
<td></td>
</tr>
<tr>
<td>$o^1$ a y x #</td>
<td>one:a three:y four:x five:# (= $o^1, s^1$)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>one:a three:y three:x five:# (= $o^1, s^2$)</td>
<td></td>
</tr>
<tr>
<td>$o^2$ b x a y #</td>
<td>two:b four:x one:a three:y five:# (= $o^2, s^3$)</td>
<td></td>
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$o^0$ and the 10 copies of $o^2$ get actually get only 1 possible completion each

$o^1$ though gets two possible completions, equally likely ones

\[ ^1 \text{the 'hidden' states are not really very hidden!} \]
brute-force EM: considering all the possible paths

On our initial settings what state sequences are possible for these observation sequences?

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<th>= (o, s)</th>
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<td>o^0</td>
<td>a x z #</td>
<td>one:a three:x four:z five: #</td>
</tr>
<tr>
<td>o^1</td>
<td>a y x #</td>
<td>one:a three:y four:x five: #</td>
</tr>
<tr>
<td>o^2</td>
<td>b x a y #</td>
<td>one:a three:y three:x five: #</td>
</tr>
<tr>
<td></td>
<td></td>
<td>two:b four:x one:a three:y five: #</td>
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o^0 and the 10 copies of o^2 get actually get only 1 possible completion each\(^1\)
o\(^1\) though gets two possible completions, equally likely ones

this is because when you look to the terms that differ between \(P(o^1, s^1)\) and \(P(o^1, s^2)\), numerically there’s no difference

\[
\begin{align*}
P(\text{four}|\text{three}) &= 0.25 & P(\text{x}|\text{four}) &= 0.5 & P(\text{five}|\text{four}) &= 0.5 \\
P(\text{three}|\text{three}) &= 0.25 & P(\text{x}|\text{three}) &= 0.5 & P(\text{five}|\text{three}) &= 0.5
\end{align*}
\]

\(^1\) the 'hidden' states are not really very hidden!
brute-force EM: conditional probs of paths

To apply (brute-force) EM, for each $o^d$, need for all possible $s$, $p(s|o^d)$ (ie. the 'responsibility' $\gamma^d(s)$). Given the forgoing we get:

\[
\begin{align*}
o^0 & : a \times z \# & s^0 : \text{one three four five} & P(s^0|o^0) = 1 \quad (= \gamma^0(s^0)) \\
o^1 & : a \ y \ x \# & s^1 : \text{one three four five} & P(s^1|o^1) = 0.5 \quad (= \gamma^1(s^1)) \\
& & s^2 : \text{one three three five} & P(s^2|o^1) = 0.5 \quad (= \gamma^1(s^2)) \\
o^2 & : b \ x \ a \ y \# & s^3 : \text{two four one three five} & P(s^3|o^2) = 1 \quad (= \gamma^2(s^3)) \\
\vdots & & \vdots & \vdots \\
o^{11} & : b \ x \ a \ y \# & s^3 : \text{two four one three five} & P(s^3|o^{11}) = 1 \quad (= \gamma^{11}(s^3))
\end{align*}
\]
A and B: expected counts and ratios

will compute *expected counts* of events relevant for A and B based on the virtual corpus of completions:

<table>
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<th>'count' of completion</th>
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<td></td>
<td>1× one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o^1</td>
<td>a y x #</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y three:x five:#</td>
</tr>
<tr>
<td>o^2</td>
<td>b x a y #</td>
</tr>
<tr>
<td></td>
<td>10× 1× two:b four:x one:a three:y five:#</td>
</tr>
</tbody>
</table>

Note although x can be generated from *four* or *three*, in o^2 only *four* is possible, so *four:*x is more frequent in this virtual corpus than it was in the fully observed corpus we started with
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Note although x can be generated from *four* or *three*, in o^2 only *four* is possible, so *four:*x is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new A and B, eg.
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<td>o^1</td>
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possible, so *four*:x is more frequent in this virtual corpus than it was in the fully
observed corpus we started with
We derive expected counts for 'state-then-state' and 'obs-with-state', then
derive new A and B, eg.

\[
A^1 \quad E(\text{three, four}) = 1.5 \quad E(\text{three}) = 12.5 \\
E(\text{three, three}) = 0.5 \\
E(\text{three, five}) = 10.5 \\
E(\text{four, five}) = 1.5 \quad E(\text{four}) = 11.5 \\
E(\text{four, one}) = 10 \\
B^1 \quad E(\text{three, x}) = 1.5 \quad E(\text{three}) = 12.5 \\
E(\text{three, y}) = 11 \\
E(\text{four, z}) = 1 \quad E(\text{four}) = 11.5 \\
E(\text{four, x}) = 10.5
\]
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<td></td>
<td>1× one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o^1</td>
<td>a y × #</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y three:x five:#</td>
</tr>
<tr>
<td>o^2</td>
<td>b × a y #</td>
</tr>
<tr>
<td></td>
<td>10× 1× two:b four:x one:a three:y five:#</td>
</tr>
</tbody>
</table>

Note although x can be generated from four or three, in o^2 only four is possible, so four:x is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new A and B, eg.

\[
\begin{align*}
A^1 \quad E(\text{three, four}) &= 1.5 \quad E(\text{three}) = 12.5 \\
E(\text{three, three}) &= 0.5 \\
E(\text{three, five}) &= 10.5 \\
E(\text{four, five}) &= 1.5 \quad E(\text{four}) = 11.5 \\
E(\text{four, one}) &= 10 \\
B^1 \quad E(\text{three, x}) &= 1.5 \quad E(\text{three}) = 12.5 \\
E(\text{three, y}) &= 11 \\
E(\text{four, z}) &= 1 \quad E(\text{four}) = 11.5 \\
E(\text{four, x}) &= 10.5
\end{align*}
\]
A and B: expected counts and ratios

will compute *expected counts* of events relevant for A and B based on the virtual corpus of completions:

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<tr>
<td>$o^0$</td>
<td>a x z # 1× one:a three:x four:z five:#</td>
</tr>
<tr>
<td>$o^1$</td>
<td>a y x # 0.5× one:a three:y four:x five:# 0.5× one:a three:y three:x five:#</td>
</tr>
<tr>
<td>$o^2$</td>
<td>b x a y # 10× 1× two:b four:x one:a three:y five:#</td>
</tr>
</tbody>
</table>

Note although $x$ can be generated from *four* or *three*, in $o^2$ only *four* is possible, so *four:*$x$ is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new $A$ and $B$, eg.

$A^1$

\[
\begin{align*}
E(\text{three}, \text{four}) &= 1.5 & E(\text{three}) &= 12.5 \\
E(\text{three, three}) &= 0.5 \\
E(\text{three, five}) &= 10.5 \\
E(\text{four, five}) &= 1.5 & E(\text{four}) &= 11.5 \\
E(\text{four, one}) &= 10
\end{align*}
\]

$B^1$

\[
\begin{align*}
E(\text{three, } x) &= 1.5 & E(\text{three}) &= 12.5 \\
E(\text{three, } y) &= 11 \\
E(\text{four, } z) &= 1 & E(\text{four}) &= 11.5 \\
E(\text{four, } x) &= 10.5
\end{align*}
\]
A and B: expected counts and ratios

will compute *expected counts* of events relevant for A and B based on the virtual corpus of completions:

<table>
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<tr>
<th>observations</th>
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<tbody>
<tr>
<td>o⁰</td>
<td>a × z #</td>
</tr>
<tr>
<td></td>
<td>1× one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o¹</td>
<td>a y × #</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y three:x five:#</td>
</tr>
<tr>
<td>o²</td>
<td>b × a y #</td>
</tr>
<tr>
<td></td>
<td>10× one:a three:y five:#</td>
</tr>
<tr>
<td></td>
<td>1× two:b four:x one:a three:y five:#</td>
</tr>
</tbody>
</table>

Note although x can be generated from *four* or *three*, in o² only *four* is possible, so *four:*x is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new A and B, eg.

\[
A¹ \quad E(three, four) = 1.5 \quad E(three) = 12.5 \\
E(three, three) = 0.5 \\
E(three, five) = 10.5 \\
E(four, five) = 1.5 \quad E(four) = 11.5 \\
E(four, one) = 10 \\
B¹ \quad E(three, x) = 1.5 \quad E(three) = 12.5 \\
E(three, y) = 11 \\
E(four, z) = 1 \quad E(four) = 11.5 \\
E(four, x) = 10.5
\]
A and B: expected counts and ratios

will compute expected counts of events relevant for A and B based on the virtual corpus of completions:

<table>
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<tr>
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<tbody>
<tr>
<td>$o^0$</td>
<td>a x z #</td>
</tr>
<tr>
<td></td>
<td>1× one:a three:x four:z five:#</td>
</tr>
<tr>
<td>$o^1$</td>
<td>a y x #</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y four:x five:z#</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>$o^2$</td>
<td>b x a y #</td>
</tr>
<tr>
<td></td>
<td>10× one:a three:y four:x one:a three:y five:#</td>
</tr>
<tr>
<td></td>
<td>1× two:b four:x one:a three:y five:#</td>
</tr>
</tbody>
</table>

Note although x can be generated from four or three, in $o^2$ only four is possible, so four:x is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new $A$ and $B$, eg.

\[
A^1 \quad E(three, four) = 1.5 \quad E(three) = 12.5 \\
E(three, three) = 0.5 \\
E(three, five) = 10.5 \\
E(four, five) = 1.5 \quad E(four) = 11.5 \\
E(four, one) = 10
\]

\[
B^1 \quad E(three, x) = 1.5 \quad E(three) = 12.5 \\
E(three, y) = 11 \\
E(four, z) = 1 \quad E(four) = 11.5 \\
E(four, x) = 10.5
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<tr>
<td>o^1</td>
<td>a y × #</td>
</tr>
<tr>
<td>o^2</td>
<td>b × a y #</td>
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Note although *x* can be generated from *four* or *three*, in o^2 only *four* is possible, so *four:*x is more frequent in this virtual corpus than it was in the fully observed corpus we started with.

We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new A and B, eg.

\[
\begin{align*}
A^1 & \quad E(\text{three}, \text{four}) = 1.5 & E(\text{three}) = 12.5 \\
& \quad E(\text{three}, \text{three}) = 0.5 \\
& \quad E(\text{three}, \text{five}) = 10.5 \\
& \quad E(\text{four}, \text{five}) = 1.5 & E(\text{four}) = 11.5 \\
& \quad E(\text{four}, \text{one}) = 10 \\
B^1 & \quad E(\text{three}, x) = 1.5 & E(\text{three}) = 12.5 \\
& \quad E(\text{three}, y) = 11 \\
& \quad E(\text{four}, z) = 1 & E(\text{four}) = 11.5 \\
& \quad E(\text{four}, x) = 10.5
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<td>o^2</td>
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We derive expected counts for 'state-then-state' and 'obs-with-state', then derive new A and B, eg.

\[ A^1 \]
\[ E(three, four) = 1.5 \quad E(three) = 12.5 \quad P(four|three) = 1.5/12.5 \]
\[ E(three, three) = 0.5 \]
\[ E(three, five) = 10.5 \]
\[ E(four, five) = 1.5 \quad E(four) = 11.5 \]
\[ E(four, one) = 10 \]

\[ B^1 \]
\[ E(three, x) = 1.5 \quad E(three) = 12.5 \]
\[ E(three, y) = 11 \]
\[ E(four, z) = 1 \quad E(four) = 11.5 \]
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& \quad E(\text{three}, \text{three}) = 0.5 \\
& \quad E(\text{three}, \text{five}) = 10.5 \\
& \quad E(\text{four}, \text{five}) = 1.5 \quad E(\text{four}) = 11.5 \\
& \quad E(\text{four}, \text{one}) = 10 \\
B^1 & \quad E(\text{three}, \text{x}) = 1.5 \quad E(\text{three}) = 12.5 \quad P(\text{x}|\text{three}) = 1.5/12.5 \\
& \quad E(\text{three}, \text{y}) = 11 \\
& \quad E(\text{four}, \text{z}) = 1 \quad E(\text{four}) = 11.5 \\
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& \quad E(\text{three}, \text{three}) = 0.5 \\
& \quad E(\text{three}, \text{five}) = 10.5 \\
& \quad E(\text{four}, \text{five}) = 1.5 \quad E(\text{four}) = 11.5 \quad P(\text{five}|\text{three}) = 10.5/12.5 \\
& \quad E(\text{four}, \text{one}) = 10 \\
\end{align*}
\]

\[
\begin{align*}
B^1 & \quad E(\text{three}, \text{x}) = 1.5 \quad E(\text{three}) = 12.5 \quad P(\text{x}|\text{three}) = 1.5/12.5 \\
& \quad E(\text{three}, \text{y}) = 11 \\
& \quad E(\text{four}, \text{z}) = 1 \quad E(\text{four}) = 11.5 \\
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<tbody>
<tr>
<td>o^0 a x z #</td>
<td>1 × one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o^1 a y x #</td>
<td>0.5× one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y three:x five:#</td>
</tr>
<tr>
<td>o^2 b x a y #</td>
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derive expected counts of 'state at start', then new π:
the virtual corpus of completions:

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</table>
| o<sup>0</sup> a × z # | 1 × one:a three:x four:z five:#{
| o<sup>1</sup> a y x # | 0.5× one:a three:y four:x five:#{
| o<sup>2</sup> b × a y # | 10× 1× two:b four:x one:a three:y five:#{

derive expected counts of 'state at start', then new π :

\[
\pi^1 \quad E(s_1 = one) = 2 \\
E(s_1 = two) = 10
\]
the virtual corpus of completions:

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<tbody>
<tr>
<td>o⁰ a x z #</td>
<td>1 × one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o¹ a y x #</td>
<td>0.5× one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5× one:a three:y three:x five:#</td>
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<tr>
<td>o² b x a y #</td>
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\[
\begin{align*}
\pi² \quad & E(s₁ = one) = 2 \\
& E(s₁ = two) = 10
\end{align*}
\]
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<tr>
<td>$o^0$ a x z #</td>
<td>$1 \times$ one:a three:x four:z five:z #</td>
</tr>
<tr>
<td>$o^1$ a y x #</td>
<td>$0.5 \times$ one:a three:y four:x five:z #</td>
</tr>
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<td>$o^2$ b x a y #</td>
<td>$10 \times$ one:a three:y three:x five:z #</td>
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<td>a × z # 1 × one:a three:x four:z five:#</td>
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<tr>
<td>o^1</td>
<td>a y x # 0.5× one:a three:y four:x five:# 0.5× one:a three:y three:x five:#</td>
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<td>o^2</td>
<td>b × a y # 10× 1× two:b four:x one:a three:y five:#</td>
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<td>1 x one:a three:x four:z five:#</td>
</tr>
<tr>
<td>o1 a y x #</td>
<td>0.5x one:a three:y four:x five:#</td>
</tr>
<tr>
<td></td>
<td>0.5x one:a three:y three:x five:#</td>
</tr>
<tr>
<td>o2 b x a y #</td>
<td>10x two:b four:x one:a three:y five:#</td>
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E(s_1 = \textit{two}) = 10
$$
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<tr>
<td>$o^0$</td>
<td>$a \times z #$</td>
</tr>
<tr>
<td>$o^1$</td>
<td>$a \ y \times #$</td>
</tr>
<tr>
<td>$o^2$</td>
<td>$b \times a \ y #$</td>
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</table>

derive expected counts of 'state at start', then new $\pi$:

$$
\begin{align*}
\pi^1 & \quad E(s_1 = \text{one}) = 2 \\
& \quad P(s_1 = \text{one}) = 2/12 \\
& \quad E(s_1 = \text{two}) = 10
\end{align*}
$$
\( \pi \): expected counts and ratios

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<td>( a \times z # )</td>
</tr>
<tr>
<td>( o^1 )</td>
<td>( a \ y \ x # )</td>
</tr>
<tr>
<td>( o^2 )</td>
<td>( b \times a \ y # )</td>
</tr>
<tr>
<td></td>
<td>( 10 \times )</td>
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derive expected counts of 'state at start', then new \( \pi \):

\[
\begin{align*}
\pi^1 & \quad E(s_1 = one) = 2 \quad P(s_1 = one) = 2/12 \\
\quad & \quad E(s_1 = two) = 10 \quad P(s_1 = two) = 10/12
\end{align*}
\]
Some worked eg of Parameter Estimation with HMMs: a fully observed case and the 'brute-force' EM unsupervised case

Outline

---

after one iteration

so after one iteration of brute force EM the new values for $\pi$ and $A$ and $B$ are somewhat different from the starting values, here’s some:

<table>
<thead>
<tr>
<th>$\pi^1$</th>
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<th>$B^1$</th>
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<tbody>
<tr>
<td>$P(\text{start} = \text{one}) = 2/12$</td>
<td>$P(\text{four}</td>
<td>\text{three}) = 1.5/12.5$</td>
</tr>
<tr>
<td>$P(\text{start} = \text{two}) = 10/12$</td>
<td>$P(\text{three}</td>
<td>\text{three}) = 0.5/12.5$</td>
</tr>
<tr>
<td>$P(\text{five}</td>
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\[ \pi^1 \]

\[ \begin{align*}
P(\text{start } = \text{one}) &= 2/12 \\
P(\text{start } = \text{two}) &= 10/12
\end{align*} \]

\[ A^1 \]

\[ \begin{align*}
P(\text{four}|\text{three}) &= 1.5/12.5 \\
P(\text{three}|\text{three}) &= 0.5/12.5 \\
P(\text{five}|\text{three}) &= 10.5/12.5 \\
P(\text{five}|\text{four}) &= 1.5/11.5
\end{align*} \]

\[ B^1 \]

\[ \begin{align*}
P(\text{x}|\text{three}) &= 1.5/12.5 \\
P(\text{x}|\text{four}) &= 10.5/11.5
\end{align*} \]

For example, concerning a $y \times \#$, whereas previously there two state sequences that could pair equally well with this, now there is an asymmetry.

\[ \frac{P(\text{one } : \text{a } \text{three } : y \text{ four } : x \text{ five } : \#)}{P(\text{one } : \text{a } \text{three } : y \text{ three } : x \text{ five } : \#)} = \frac{1.5/12.5 \times 10.5/11.5 \times 1.5/11.5}{0.5/12.5 \times 1.5/12.5 \times 10.5/12.5} = 0.0143 \]

\[ \frac{P(\text{one } : \text{a } \text{three } : y \text{ three } : x \text{ five } : \#)}{P(\text{one } : \text{a } \text{three } : y \text{ four } : x \text{ five } : \#)} = \frac{0.5/12.5 \times 1.5/12.5 \times 10.5/12.5}{0.5/12.5 \times 1.5/12.5 \times 10.5/12.5} = 0.0040 \]
after one iteration

so after one iteration of brute force EM the new values for $\pi$ and $A$ and $B$ are somewhat different from the starting values, here’s some:

$$
\begin{align*}
\pi^1 & \\
P(\text{start = one}) &= 2/12 & P(\text{four | three}) &= 1.5/12.5 & P(\text{x | three}) &= 1.5/12.5 \\
P(\text{start = two}) &= 10/12 & P(\text{three | three}) &= 0.5/12.5 & P(\text{x | four}) &= 10.5/11.5 \\
& & P(\text{five | three}) &= 10.5/12.5 & & \\
& & P(\text{five | four}) &= 1.5/11.5 & \\
\end{align*}
$$

For example, concerning $a\ y\ x\ \#$, whereas previously there two state sequences that could pair equally well with this, now there is an asymmetry.

$$
\frac{P(\text{one : a three : y four : x five : #})}{P(\text{one : a three : y three : x five : #})} = \frac{1.5/12.5 \times 10.5/11.5 \times 1.5/11.5}{0.5/12.5 \times 1.5/12.5 \times 1.5/12.5} = 0.0143
$$

So one round of EM has definitely changed the probabilities; really all this should be every expectable given previous study of EM applied in (i) the coin tossing scenario – hidden A vs B choice (ii) the IBM Model 1 scenario – hidden alignment. We 'just' 'have a different hidden variable here – the hidden state sequence.
after one iteration

so after one iteration of brute force EM the new values for $\pi$ and $A$ and $B$ are somewhat different from the starting values, here's some:

\[
\begin{align*}
\pi^1 & \\
\text{P(start = one)} &= 2/12 \\
\text{P(start = two)} &= 10/12 \\

A^1 & \\
\text{P(four|three)} &= 1.5/12.5 \\
\text{P(three|three)} &= 0.5/12.5 \\
\text{P(five|three)} &= 10.5/12.5 \\
\text{P(five|four)} &= 1.5/11.5 \\

B^1 & \\
\text{P(x|three)} &= 1.5/12.5 \\
\text{P(x|four)} &= 10.5/11.5
\end{align*}
\]

For example, concerning $a y x \#$, whereas previously there two state sequences that could pair equally well with this, now there is an asymmetry.

\[
\frac{P(\text{one : a three : y four : x five : #})}{P(\text{one : a three : y three : x five : #})} = \frac{1.5/12.5 \times 10.5/11.5 \times 1.5/11.5}{0.5/12.5 \times 1.5/12.5 \times 10.5/12.5} = 0.0143
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