1. Consider the following equations (alternative conditions for independence)

(i) \( P(A \land B) = P(A) \times P(B) \)
(ii) \( P(A|B) = P(A) \)

show that (i) implies (ii), and also that (ii) implies (i)

Answer
showing (i) \( \Rightarrow \) (ii): suppose \( P(A \land B) = P(A) \times P(B) \). By definition \( P(A|B) = \frac{P(A \land B)}{P(B)} \), hence \( P(A|B) = \frac{P(A) \times P(B)}{P(B)} = P(A) \).

showing (ii) \( \Rightarrow \) (i): suppose \( P(A|B) = P(A) \). By the product rule \( P(A|B) \times P(B) = P(A \land B) \), hence \( P(A \times P(B) = P(A \land B) \)

2. Suppose \textit{Harry Potter} plays for the \textit{Gryffindor} team, and suppose he ‘often’ catches the so-called \textit{Golden Snitch}, and also that ‘often’ Gryffindor win. Suppose the numbers actually are as follows

<table>
<thead>
<tr>
<th></th>
<th>( gw )</th>
<th>( \neg gw )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ps )</td>
<td>28</td>
<td>2</td>
</tr>
<tr>
<td>( \neg ps )</td>
<td>140</td>
<td>30</td>
</tr>
</tbody>
</table>

\( ps \) means ‘Harry Potter caught the Golden Snitch’
\( \neg ps \) means ‘Gryffindor won’

(a) Calculate \( P(gw | ps) \), the conditional prob that Gryffindor won given that Harry Potter caught the Golden Snitch and also indicate which counts are irrelevant to this calculation

Answer
\( P(gw | ps) = \frac{28}{30} = \frac{14}{15} \), because can be calculated from \( \frac{\text{#}(ps, gw)}{\text{#}ps} \)
the counts in the \( \neg ps \) row are irrelevant, ie. 140 and 30.

Remark you could calculate \( P(ps, gw) \) and \( P(ps) \) and take their ratio, strictly following the defin. of conditional probability but doing so involves you in more work than necessary. Same applies in next part.

(b) Calculate \( P(ps | gw) \), the conditional prob that Harry Potter caught the Golden Snitch given that Gryffindor won and also indicate which counts are irrelevant to this calculation

Answer
\( P(ps | gw) = \frac{28}{168} = \frac{1}{6} \), because can be calculated from \( \frac{\text{#}(ps, gw)}{\text{#}gw} \)
the counts in the \( \neg gw \) column are irrelevant, ie. 2 and 30.

3. A sound clip from YouTube may or may not have been produced by Victor Meldrew. A clip may or may not contain the phrase ‘I don’t believe it’. 
You hear a clip coming from YouTube containing the phrase ‘I don’t believe it’ and want to work out the probability that the speaker was Victor Meldrew.

Formalize with 2 discrete variables

- discrete Speaker, values in {'Victor Meldrew', 'Other'}
- discrete DBI, values in {true, false}, for whether a sound clip contains the phrase ‘I don’t believe it’

Let $v_{mel}$ stand for $\text{Speaker} = \text{Victor Meldrew}$, $dbi$ stands for $\text{DBI} = \text{true}$

(a) Work out which of $v_{mel}$ or $\neg v_{mel}$ is likelier, given $dbi$, supposing the probabilities $p(v_{mel}) = 0.01$, $p(dbi|v_{mel}) = 0.95$, $p(dbi|\neg v_{mel}) = 0.01$

**ANSWER** $\neg v_{mel}$ is likelier.

working it out assisted by R (note > is the R prompt and not something that you type):

```r
> vmel = 0.01
> dbi_given_vmel = 0.95
> dbi_given_other = 0.01
> dbi_and_vmel = vmel * dbi_given_vmel
> dbi_and_other = (1 - vmel) * dbi_given_other
> dbi_and_vmel
[1] 0.0095
> dbi_and_other
[1] 0.0099
```

so $\neg v_{mel}$ is likelier.

*Note* though not at all necessary to answer the question, one can work a little harder to to calculate the conditional probs, normalising by the sums

```r
> vmel_given_dbi = dbi_and_vmel/(dbi_and_vmel + dbi_and_other)
> vmel_given_dbi
[1] 0.4896907
> other_given_dbi = dbi_and_other/(dbi_and_vmel + dbi_and_other)
> other_given_dbi
[1] 0.5103093
```

so $P(\neg v_{mel}| omg) = 0.51$. As emphasized in lectures, these manoeuvres are *not* necessary to simply answer the categorisation question.

(b) Do the same assuming $p(v_{mel}) = 0.15$, $p(dbi|v_{mel}) = 0.95$, $p(dbi|\neg v_{mel}) = 0.01$

**ANSWER** $v_{mel}$ is likelier

the corresponding outcomes in R will be

```
[1] "dbi_and_vmel" "0.1425"
[1] "dbi_and_other" "0.0085"
[1] "vmel_given_dbi" "0.943708609271523"
[1] "other_given_dbi" "0.0562913907284768"
```
Again note the categorisation can be settled by comparing the joint probs 0.1424 vs. 0.0085, though the conditional probs can be subsequently worked out, giving that \( P(\text{vmel} \mid \text{dbi}) = 0.94 \)

You can get the above outputs in R, if you enter the following function definition

```r
f <- function() {
  dbi_and_vmel = vmel * dbi_given_vmel;
  dbi_and_other = (1 - vmel) * dbi_given_other;
  print(c("dbi_and_vmel",dbi_and_vmel));
  print(c("dbi_and_other",dbi_and_other));
  vmel_given_dbi = dbi_and_vmel/(dbi_and_vmel + dbi_and_other);
  other_given_dbi = dbi_and_other/(dbi_and_vmel + dbi_and_other);
  print(c("vmel_given_dbi",vmel_given_dbi));
  print(c("other_given_dbi",other_given_dbi));
}
```

and then after making sure that \( \text{vmel}, \text{dbi}_\text{given}_\text{vmel} \) and \( \text{dbi}_\text{given}_\text{other} \) have the right values, just execute \( f() \)

(c) Do the same assuming \( p(\text{vmel}) = 0.01, p(\text{dbi} \mid \text{vmel}) = 0.95, p(\text{dbi} \mid \neg \text{vmel}) = 0.001 \)

**ANSWER**

\( \text{vmel} \) is likelier outcomes via R:

```
[1] "dbi_and_vmel" "0.0095"
[1] "dbi_and_other" "0.00099"
[1] "vmel_given_dbi" "0.905624404194471"
[1] "other_given_dbi" "0.094375595805529"
```

*Note* same remarks apply re. joint and conditional

4. Consider someone who lives in a basement flat. Sometimes it is quite noisy in the flat, and sometimes not. Sometimes it is rather cool in the flat, and sometimes not. Let \( \text{noisy} \) be a variable indicating whether it is rather noisy or not, on a given day, and let \( \text{cool} \) be a variable indicating whether it is rather cool or not.

Consider the frequency table

<table>
<thead>
<tr>
<th></th>
<th>noisy :+</th>
<th>noisy :-</th>
</tr>
</thead>
<tbody>
<tr>
<td>cool :+</td>
<td>62</td>
<td>108</td>
</tr>
<tr>
<td>cool :-</td>
<td>38</td>
<td>292</td>
</tr>
</tbody>
</table>

find \( p(\text{cool} : +) \) and \( p(\text{cool} : + \mid \text{noisy} : +) \)

and conclude from this whether or not \( \text{cool} : + \) is independent of \( \text{noisy} : + \)

**ANSWER**

\[
\text{find } p(\text{cool} : +) = \frac{62+108}{300} = 0.34, p(\text{cool} : + \mid \text{noisy} : +) = \frac{62}{62+38} = 0.62. \text{ These are not equal, so } \text{cool} : + \text{ is not independent of } \text{noisy} : +
\]
> cool = (62+108)/(62+108+38+292)
> cool
[1] 0.34
> cool_given_noisy = 62/(62+38)
> cool_given_noisy
[1] 0.62

5. Unknown to the occupant of the flat there is a ventilator fixture in the wall which can be opened and shut to let air from the street outside in, or keep it out. Unknown to the occupant a pet cat plays about with this at night-time, sometimes leaving it open and sometimes leaving it shut. The table (1) concerns 500 days. The two tables below split these into a group of 100 days where the cat has left the ventilator open (2), and 400 days where the cat has left it shut (3)

\[
\begin{array}{ccc}
\text{open} : + & \text{noisy} : + & \text{noisy} : - \\
\text{cool} : + & 54 & 36 \\
\text{cool} : - & 6 & 4 \\
\end{array}
\]  

(2)

\[
\begin{array}{ccc}
\text{open} : - & \text{noisy} : + & \text{noisy} : - \\
\text{cool} : + & 8 & 72 \\
\text{cool} : - & 32 & 288 \\
\end{array}
\]  

(3)

With reference to the table (2), find \(p(\text{cool} : + | \text{open} : +)\) and \(p(\text{cool} : + | \text{open} : +, \text{noisy} : +)\) and conclude from this whether or not \(\text{cool} : +\) is conditionally independent of \(\text{noisy} : +\) given \(\text{open} : +\).

With reference to the table (3), find \(p(\text{cool} : + | \text{open} : -)\) and \(p(\text{cool} : + | \text{open} : -, \text{noisy} : +)\) and conclude from this whether or not \(\text{cool} : +\) is conditionally independent of \(\text{noisy} : +\) given \(\text{open} : -\).

**ANSWER**

\[p(\text{cool} : + | \text{open} : +) = 90/100 = 9/10\]
\[p(\text{cool} : + | \text{open} : +, \text{noisy} : +) = 54/60 = 9/10\]

since these are the same \(\text{cool} : +\) is conditionally independent of \(\text{noisy} : +\) given \(\text{open} : +\)

\[p(\text{cool} : + | \text{open} : -) = 80/400 = 1/5\]
\[p(\text{cool} : + | \text{open} : -, \text{noisy} : +) = 8/40 = 1/5\]

since these are the same \(\text{cool} : +\) is conditionally independent of \(\text{noisy} : +\) given \(\text{open} : -\)