FUZZY CONTROL:
Mamdani & Takagi-Sugeno Controllers

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The term *control* is generally defined as a mechanism used to guide or regulate the operation of a machine, apparatus or constellations of machines and apparatus.
FUZZY CONTROL

Control Theory?

• In Takagi–Sugeno (T–S) fuzzy model, the state space of a nonlinear system is divided into different fuzzy regions with a local linear model being used in each region. The overall model output is obtained by de-fuzzification using the center of gravity (COG) method.
FUZZY CONTROL

Control Theory?

An Input/Output Relationship
FUZZY CONTROL

Control Theory?

An Input/Output Relationship IDENTIFIED using Fuzzy Logic
FUZZY CONTROL

Control Theory?

An Input/Output Relationship IDENTIFIED using Fuzzy Logic

\[
\text{if } x_1 \text{ is } 0 \text{ to } 7 \quad \text{then } y = 0.6x + 2
\]

\[
\text{if } x_1 \text{ is } 4 \text{ to } 10 \quad \text{then } y = 0.2x + 9.
\]
Fuzzy Control

Control Theory?

• Typically, rules contain membership functions for both antecedents and consequent.

• Argument is that the consequent membership function can be simplified – this argument is based on a heuristic that operators in a control environment divide the variable space (say, error, change in error and change in control) into PARTITIONS;

• Within each partition the output variable is a simple, often linear function of the input variables and not membership functions.
Mamdani Controllers: Rules contain membership functions for both antecedents and consequent:

If $e(k)$ is positive($e$) and $\Delta e(k)$ is positive($\Delta e$) then $\Delta u(k)$ is positive($\Delta u$)
Takagi-Sugeno Controllers: Rules contain membership functions for antecedents and linear functions in the consequent.

If $e(k)$ is positive(e) and $\Delta e(k)$ is positive($\Delta e$) then $\Delta u(k) = \alpha e(k) + \beta \Delta e(k) + \delta$; where, $\alpha$, $\beta$ and $\delta$ are obtained from empirical observations by relating the behaviour of the errors and change in errors over a fixed range of changes in control.
According to Yager and Filev, ‘a known disadvantage of the linguistic modules is that they do not contain in an explicit form the objective knowledge about the system if such knowledge cannot be expressed and/or incorporated into fuzzy set framework' (1994:192).

Typically, such knowledge is available often: for example in physical systems this kind of knowledge is available in the form of general conditions imposed on the system through conservation laws, including energy mass or momentum balance, or through limitations imposed on the values of physical constants.
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

Tomohiro Takagi and Michio Sugeno recognised two important points:

1. Complex technological processes may be described in terms of interacting, yet simpler sub processes. This is the mathematical equivalent of fitting a piece-wise linear equation to a complex curve.

2. The output variable(s) of a complex physical system, e.g. complex in the sense it can take a number of input variables to produce one or more output variable, can be related to the system's input variable in a linear manner provided the output space can be subdivided into a number of distinct regions.

Takagi-Sugeno fuzzy models have been widely used to identify the structures and parameters of unknown or partially known plants, and to control nonlinear systems.
Mamdani style inference:

The Good News: This method is regarded widely ‘for capturing expert knowledge’ and facilitates an intuitively-plausible description of knowledge;

The Bad News: This method involves the computation of a two-dimensional shape by summing, or more accurately integrating across a continuously varying function. The computation can be expensive.
Mamdani style inference:
The Bad News: This method involves the computation of a two-dimensional shape by summing, or more accurately integrating across a continuously varying function. The computation can be expensive.

For every rule we have to find the membership functions for the linguistic variables in the antecedents and the consequents;

For every rule we have to compute, during the inference, composition and defuzzification process the membership functions for the consequents;

Given the non-linear relationship between the inputs and the output, it is not easy to identify the membership functions for the linguistic variables in the consequent
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

Literature on conventional control systems has suggested that a complex non-linear system can be described as a collection of subsystems that were combined based on a logical (Boolean) switching system function.

In realistic situations such disjoint (crisp) decomposition is impossible, due to the inherent lack of natural region boundaries in the system, and also due to the fragmentary nature of available knowledge about the system.
Takagi and Sugeno (1985) have argued that in order to develop a generic and simple mathematical tool for computing fuzzy implications one needs to look at a fuzzy partition of fuzzy input space.

In each fuzzy subspace a linear input-output relation is formed. The output of fuzzy reasoning is given by the values inferred by some implications that were applied to an input.
Takagi and Sugeno have described a fuzzy implication \( R \) as:

\[
R: \text{ if } (x_1 \text{ is } \mu_A(x_1), \ldots \ x_k \text{ is } \mu_A(x_k)) \text{ then } y = g(x_1, \ldots, x_k)
\]
A solution for the coefficients of the consequent in TSK Systems

Rule 1: \( x \) is \( \mu_1(x) \) THEN \( y = p_0 + p_1 x \)

**COMPOSITION**

\[
y = \frac{\mu_1(x) \cdot [p_0 + p_1 x]}{\mu_1(x)}
\]

There are two unknowns: \( p_0 \) and \( p_1 \). So we need two simultaneous equations for two values of \( x \), say \( x_1 \) and \( x_2 \), and two values of \( y - y_1 \) and \( y - y_2 \).

\[
y_1 = p_0 + p_1 x_1 \\
y_2 = p_0 + p_1 x_2 \\
p_1 = \frac{y_1 - y_2}{x_1 - x_2};
\]

\[
p_0 = y_1 - x_1 \cdot \frac{y_1 - y_2}{x_1 - x_2}
\]
A solution for the coefficients of the consequent in TSK Systems

Consider a two rule system:

Rule\textsuperscript{1} :  \( x \) is \( \mu_1(x) \) \( \text{THEN} \ y = p_{01} + p_{11}x \)

Rule\textsuperscript{2} :  \( x \) is \( \mu_2(x) \) \( \text{THEN} \ y = p_{02} + p_{12}x \)

\textit{COMPOSITION}

\[
y = \frac{\mu_1(x) \ast [p_{01} + p_{11}x] + \mu_2(x) \ast [p_{02} + p_{12}x]}{\mu_1(x) + \mu_2(x)}
\]

\[
\hat{\mu}_1(x) = \frac{\mu_1(x) \ast [p_{01} + p_{11}x] + \mu_2(x) \ast [p_{02} + p_{12}x]}{\mu_1(x) + \mu_2(x)}
\]

\[
\therefore \ y = \hat{\mu}_1(x) \ast [p_{01} + p_{11}x] + \hat{\mu}_2(x) \ast [p_{02} + p_{12}x]
\]
A solution for the coefficients of the consequent in TSK Systems

There are 4 unknowns $p_{01}, p_{11}, p_{02}, p_{12}$ so we need 4 equations. And, these can be obtained from 4 observations comprising 4 different values of $x$ and $y$.

$$y = \hat{\mu}_1(x) \cdot [p_{01} + p_{11}x] + \hat{\mu}_2(x) \cdot [p_{02} + p_{12}x]$$

*Four values of $x = x_1, x_2, x_3, x_4$ and for each $x$, will have a value*

$$y_1 = \hat{\mu}_1(x_1) \cdot p_{01} + \hat{\mu}_2(x_1) \cdot p_{02} + \hat{\mu}_1(x_1) \cdot x_1 \cdot p_{11} + \hat{\mu}_2(x_1) \cdot x_1 \cdot p_{11}$$

*.................................*

$$y_4 = \hat{\mu}_1(x_4) \cdot p_{01} + \hat{\mu}_2(x_4) \cdot p_{02} + \hat{\mu}_1(x_4) \cdot x_4 \cdot p_{11} + \hat{\mu}_2(x_4) \cdot x_4 \cdot p_{11}$$
A solution for the coefficients of the consequent in TSK Systems

There are 4 unknowns $p_{01}, p_{11}, p_{02}, p_{12}$ so we need 4 equations. And, these can be obtained from 4 observations comprising 4 different values of and 4.

\[ \therefore y = \hat{\mu}_1(x)[p_{01} + p_{11}x] + \hat{\mu}_2(x)[p_{02} + p_{12}x] \]

Four values of \( x = x_1, x_2, x_3, x_4 \) and for each \( x \), will have a value

\[ y_1 = \hat{\mu}_1(x_1) \times p_{01} + \hat{\mu}_2(x_1) \times p_{02} + \hat{\mu}_1(x_1) \times x_1 \times p_{11} + \hat{\mu}_2(x_1) \times x_1 \times p_{12} \]

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4
\end{bmatrix} =
\begin{bmatrix}
  \hat{\mu}_1(x_1) & \hat{\mu}_2(x_1) & \hat{\mu}_2(x_1) & \hat{\mu}_1(x_1) \times x_1 + \hat{\mu}_2(x_1) \times x_1 & p_{01} \\
  \hat{\mu}_1(x_2) & \hat{\mu}_2(x_2) & \hat{\mu}_2(x_2) & \hat{\mu}_1(x_2) \times x_1 + \hat{\mu}_2(x_2) \times x_2 & p_{02} \\
  \hat{\mu}_1(x_3) & \hat{\mu}_2(x_3) & \hat{\mu}_2(x_3) & \hat{\mu}_1(x_3) \times x_3 + \hat{\mu}_2(x_3) \times x_3 & p_{11} \\
  \hat{\mu}_1(x_4) & \hat{\mu}_2(x_4) & \hat{\mu}_2(x_4) & \hat{\mu}_1(x_4) \times x_4 + \hat{\mu}_2(x_4) \times x_4 & p_{21}
\end{bmatrix}
\]
There are 4 unknowns $p_{01}, p_{11}, p_{02}, p_{12}$, so we need 4 equations.

$$
\begin{align*}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
&= 
\begin{bmatrix}
\hat{\mu}_1(x_1) \\
\hat{\mu}_2(x_1) \\
\hat{\mu}_1(x_2) \\
\hat{\mu}_2(x_2) \\
\hat{\mu}_1(x_3) \\
\hat{\mu}_2(x_3) \\
\hat{\mu}_1(x_4) \\
\hat{\mu}_2(x_4)
\end{bmatrix}
\begin{bmatrix}
\mu_1(x_1) \\
\mu_2(x_1) \\
\mu_1(x_2) \\
\mu_2(x_2) \\
\mu_1(x_3) \\
\mu_2(x_3) \\
\mu_1(x_4) \\
\mu_2(x_4)
\end{bmatrix}
\begin{bmatrix}
0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
1 \ 0 \ 0 \ 0 \\
0 \ 1 \ 0 \ 0 \\
0 \ 0 \ 1 \ 0 \\
0 \ 0 \ 0 \ 1 \\
1 \ 1 \ 1 \ 1 \\
1 \ 1 \ 1 \ 1
\end{bmatrix}
\begin{bmatrix}
p_{01} \\
p_{02} \\
p_{11} \\
p_{21}
\end{bmatrix}

\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix}
&= 
\begin{bmatrix}
\mu_1(x_1) \\
\mu_2(x_1) \\
\mu_1(x_2) \\
\mu_2(x_2) \\
\mu_1(x_3) \\
\mu_2(x_3) \\
\mu_1(x_4) \\
\mu_2(x_4)
\end{bmatrix}
\begin{bmatrix}
\mu_1(x_1) \\
\mu_2(x_1) \\
\mu_1(x_2) \\
\mu_2(x_2) \\
\mu_1(x_3) \\
\mu_2(x_3) \\
\mu_1(x_4) \\
\mu_2(x_4)
\end{bmatrix}
\begin{bmatrix}
0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
1 \ 0 \ 0 \ 0 \\
0 \ 1 \ 0 \ 0 \\
0 \ 0 \ 1 \ 0 \\
0 \ 0 \ 0 \ 1 \\
1 \ 1 \ 1 \ 1 \\
1 \ 1 \ 1 \ 1
\end{bmatrix}
\begin{bmatrix}
p_{01} \\
p_{02} \\
p_{11} \\
p_{21}
\end{bmatrix}

\begin{bmatrix}
[p] \\
[Y]
\end{bmatrix}
&= 
\begin{bmatrix}
[X]^T[X]^{-1} \\
[X]^T
\end{bmatrix}
\begin{bmatrix}
[p] \\
[Y]
\end{bmatrix}

\begin{bmatrix}
[p] \\
[Y]
\end{bmatrix}
\end{align*}
\]
A solution for the coefficients of the consequent in TSK Systems

Takagi and Sugeno have (a) generalised the method to an \( n \)-rule, \( m \)-parameter system; and (b) claim that ‘this method of identification enables us to obtain just the same parameters as the original system, if we have a sufficient number of noiseless output data for identification’ (Takagi and Sugeno, 1985:119).

\[
\begin{bmatrix}
  y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix} =
\begin{bmatrix}
  \hat{\mu}_1(x_1) & \hat{\mu}_2(x_1) & \hat{\mu}_1(x_1) * x_1 & \hat{\mu}_2(x_1) * x_1 \\
  \hat{\mu}_1(x_2) & \hat{\mu}_2(x_2) & \hat{\mu}_1(x_2) * x_2 & \hat{\mu}_2(x_2) * x_2 \\
  \hat{\mu}_1(x_3) & \hat{\mu}_2(x_3) & \hat{\mu}_1(x_3) * x_3 & \hat{\mu}_2(x_3) * x_3 \\
  \hat{\mu}_1(x_4) & \hat{\mu}_2(x_4) & \hat{\mu}_1(x_4) * x_4 & \hat{\mu}_2(x_4) * x_4 \\
\end{bmatrix}
\begin{bmatrix}
p_{01} \\
p_{02} \\
p_{11} \\
p_{21} \\
\end{bmatrix}
\]

\[
[Y] = [X]^T [P]
\]

\[
[P] = [X^T X]^{-1} [X]^T [Y]
\]
A solution for the coefficients of the consequent in TSK Systems

In order to determine the values of the parameters $p$ in the consequents, one solves the LINEAR system of algebraic equations and tries to minimize the difference between the ACTUAL output of the system ($Y$) and the simulation $[X]^T[P]$:

$$[Y] - [X]^T[P] \leq \varepsilon$$
A solution for the coefficients of the consequent in TSK Systems

In order to determine the values of the parameters $p$ in the consequents, one solves the LINEAR system of algebraic equations and tries to minimize the difference between the ACTUAL output of the system ($Y$) and the simulation $[X]^T[P]$:

$$[Y] - [X]^T[P] \leq \varepsilon$$
Takagi and Sugeno have described a fuzzy implication $R$ as:

$$R: \text{if } (x_1 \text{ is } \mu_A(x_1), \ldots, x_k \text{ is } \mu_A(x_k)) \text{ then } y = g(x_1, \ldots, x_k), \text{ where:}$$

A zero order Takagi-Sugeno Model will be given as

$$R: \text{if } (x_1 \text{ is } \mu_A(x_1), \ldots, x_k \text{ is } \mu_A(x_k)) \text{ then } y = k$$
Consider the problem of controlling an air-conditioner (again). The rules that are used to control the air-conditioner can be expressed as a cross product:

\[
\text{CONTROL} = \text{TEMP} \times \text{SPEED}
\]
Fuzzy Logic & Fuzzy Systems

Knowledge Representation & Reasoning: The Air-conditioner Example

The rules can be expressed as a cross product of two term sets: Temperature and Speed.

\[
\text{CONTROL} = \text{TEMP} \times \text{SPEED}
\]

Where the set of linguistic values of the term sets is given as

\[
\begin{align*}
\text{TEMP} &= \text{COLD} + \text{COOL} + \text{PLEASANT} + \text{WARM} + \text{HOT} \\
\text{SPEED} &= \text{MINIMAL} + \text{SLOW} + \text{MEDIUM} + \text{FAST} + \text{BLAST}
\end{align*}
\]
Knowledge Representation & Reasoning: The Air-conditioner Example

A Mamdani Controller

Recall that the rules governing the air-conditioner are as follows:

RULE#1: IF **TEMP** is COLD THEN **SPEED** is MINIMAL

RULE#2: IF **TEMP** is COOL THEN **SPEED** is SLOW

RULE#3: IF **TEMP** is PLEASENT THEN **SPEED** is MEDIUM

RULE#4: IF **TEMP** is WARM THEN **SPEED** is FAST

RULE#5: IF **TEMP** is HOT THEN **SPEED** is BLAST
A Zero-order Takagi-Sugeno Controller

Recall that the rules governing the air-conditioner are as follows:

RULE#1: IF TEMP is COLD THEN SPEED = $k_1$

RULE#2: IF TEMP is COOL THEN SPEED = $k_2$

RULE#3: IF TEMP is PLEASENT THEN SPEED = $k_3$

RULE#4: IF TEMP is WARM THEN SPEED = $k_4$

RULE#5: IF TEMP is HOT THEN SPEED = $k_5$
A First-order Takagi-Sugeno Controller

Recall that the rules governing the air-conditioner are as follows:

RULE#1:  IF  TEMP is COLD  THEN  SPEED =\ j_1 + k_1 * T

RULE#2:  IF  TEMP is COOL  THEN  SPEED =\ j_2 + k_2 * T

RULE#3:  IF  TEMP is PLEASENT  THEN  SPEED =\ k_3

RULE#4:  IF  TEMP is WARM  THEN  SPEED =\ j_4 + k_4 * T

RULE#5:  IF  TEMP is HOT  THEN  SPEED =\ k_5
FUZZY LOGIC & FUZZY SYSTEMS

Knowledge Representation & Reasoning: The Air-conditioner Example

Temperature Fuzzy Sets

- Cold
- Cool
- Pleasant
- Warm
- Hot

Temperature Degrees C

Truth Value
Knowledge Representation & Reasoning: The Air-conditioner Example

The analytically expressed membership for the reference fuzzy subsets for the temperature are:

' COLD '  
\[ \mu_{COLD}(T) = -\frac{T}{10} + 1 \quad 0 \leq T \leq 10 \]

' COOL '  
\[ \mu_{COOL}(T) = \frac{T}{12} - 0.5 \quad 0 \leq T \leq 12.5 \]
\[ \mu_{COOL}(T) = -\frac{T}{5} + 3.5 \quad 12.5 \leq T \leq 17.5 \]

' PLEASANT '  
\[ \mu_{PLEA}(T) = \frac{T}{2.5} - 6 \quad 15 \leq T \leq 17.5 \]
\[ \mu_{PLEA}(T) = -\frac{T}{2.5} + 8 \quad 17.5 \leq T \leq 20 \]

' WARM '  
\[ \mu_{WARM}(T) = \frac{T}{5} - 3.5 \quad 17.5 \leq T \leq 22.5 \]
\[ \mu_{WARM}(T) = -\frac{T}{5} - 5.5 \quad 22.5 \leq T \leq 27.5 \]

' HOT '  
\[ \mu_{HOT}(T) = \frac{T}{2.5} - 11 \quad 25 \leq T \leq 30 \]
\[ \mu_{HOT}(T) = 1 \quad T \geq 30 \]
Knowledge Representation & Reasoning: The Air-conditioner Example

For FLC of Mamdani type
Knowledge Representation & Reasoning: The Air-conditioner Example

The zero-order speed control just takes one SINGLETON value at fixed values of the velocity; for all other values the membership function is defined as zero.

\[
\begin{align*}
'MINIMAL' & : \mu_{MINIMAL}(V) = 1 & V = 0; \\
'SLOW' & : \mu_{SLOW}(V) = 1 & V = 30 \\
'MEDIUM' & : \mu_{MED}(V) = 1 & V = 50 \\
'FAST' & : \mu_{FAST}(V) = 1 & V = 70 \\
'BLAST' & : \mu_{BLAST}(V) = 1 & V = 100
\end{align*}
\]
Knowledge Representation & Reasoning: The Air-conditioner Example

Let the temperature be 5 degrees centigrade:

**Fuzzification**: 5 degrees means that it can be COOL and COLD;

**Inference**: Rules 1 and 2 will fire:

**Composition**:

The temperature is ‘COLD’ with a truth value of $\mu_{\text{COLD}} = 0.5 \rightarrow$ the SPEED will be $k_1$

The temperature is ‘COOL’ with a truth value of $\mu_{\text{COOL}} = 0.5 \rightarrow$ the SPEED will be $k_2$

‘DEFUZZIFICATION’: CONTROL speed is

$$(\mu_{\text{COLD}} \cdot k_1 + \mu_{\text{COOL}} \cdot k_2)/(\mu_{\text{COLD}} + \mu_{\text{COOL}}) = (0.5 \cdot 0 + 0.5 \cdot 30)/(0.5 + 0.5) = 15 \text{ RPM}$$
Fuzzy Logic & Fuzzy Systems

Knowledge Representation & Reasoning: The Air-conditioner Example

Zero Order Takagi Sugeno Controller

Membership Function vs Speed

- MINIMAL
- SLOW
- MEDIUM
- FAST
- BLAST
**Fuzzy Logic & Fuzzy Systems**

**Knowledge Representation & Reasoning: The Air-conditioner Example**

**Fuzzification:** Consider that the temperature is 16°C and we want our knowledge base to compute the speed. The fuzzification of the crisp temperature gives the following membership for the Temperature fuzzy set:

<table>
<thead>
<tr>
<th>Temp=16</th>
<th>μ_COLD</th>
<th>μ_COOL</th>
<th>μ_PLEASANT</th>
<th>μ_WARM</th>
<th>μ_HOT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.3</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Fire Rule (#)**

<table>
<thead>
<tr>
<th>yes/no</th>
<th>(#1)</th>
<th>(#2)</th>
<th>(#3)</th>
<th>(#4)</th>
<th>(#5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>38</td>
</tr>
</tbody>
</table>

The table shows the membership values for the different temperature ranges and the corresponding action (yes or no) for each rule.
Fuzzy Logic & Fuzzy Systems

Knowledge Representation & Reasoning: The Air-conditioner Example

**Inference:** Consider that the temperature is 16°C and we want our knowledge base to compute the speed. Rule #2 & 3 are firing and are essentially the fuzzy patches made out of the cross products of

COOL x SLOW
PLEASANT x MEDIUM
COMPOSITION: The COOL and PLEASANT sets have an output of 0.3 and 0.4 respectively. The singleton values for SLOW and MEDIUM have to be given an alpha-level cut for these output values respectively:
DEFUZZIFICATION: The problem of finding a single, crisp value is no longer a problem for a Takagi-Sugeno controller. All we need is the weighted average of the singleton values of SLOW & MEDIUM.

Recall the Centre of Area computation for the Mamdani controller

\[
\eta = \frac{1}{\sum_{x_1, \ldots, x_n} \mu_{\text{output}}(y)} \sum_{y} y \mu_{x_1, \ldots, x_n}(y)
\]
DEFUZZIFICATION: For Takagi-Sugeno, the computation for $\eta$ is restricted to the singleton values of the SPEED linguistic variable - we do not need to sum over all values of the variable $y$:

$$\eta = \frac{\sum_y \mu_{output}^{\text{output}}(y)}{\sum \mu_{output}^{x_1 \ldots x_n}(y)}$$

$$= \frac{\sum \text{SLOW} \ast \mu_{x'}^{\text{SLOW}}(.) + \text{MEDIUM} \ast \mu_{x''}^{\text{MEDIUM}}(.)}{\mu_{x'}^{\text{SLOW}}(.) + \mu_{x''}^{\text{MEDIUM}}(.)}$$
**Fuzzy Logic & Fuzzy Systems**

Knowledge Representation & Reasoning: The Air-conditioner Example

**Defuzzification:** For Takagi-Sugeno, the computation for \( \eta \) is restricted to the singleton values of the SPEED linguistic variable – we do not need to sum over all values of the variable \( y \):

\[
\eta = \frac{\sum \text{SLOW} \cdot \mu_{x', \text{SLOW}}(.) + \text{MEDIUM} \cdot \mu_{x', \text{MEDIUM}}(.)}{\mu_{x', \text{SLOW}}(.) + \mu_{x', \text{MEDIUM}}(.)}
\]

\[
= \frac{(0.3 \cdot 30 + 0.4 \cdot 50)}{(0.3 + 0.4)} = 41.42857
\]
DEFUZZIFICATION: Recall the case of the Mamdani equivalent of the fuzzy air-conditioner – where we had fuzzy sets for the linguistic variables *SLOW* and *MEDIUM*: The ‘Centre of Area’ (COA) computations involved a weighted sum over all values of speed between 12.5 and 57.5 RPM: in the Takagi-Sugeno case we only had to consider values for speeds 30 RPM and 50 RPM.
**DEFUZZIFICATION:** Recall the case of the Mamdani equivalent of the fuzzy air-conditioner – where we had fuzzy sets for the linguistic variables *SLOW* and *MEDIUM*. The ‘Centre of Area’ (COA) computations involved a weighted sum over all values of speed between 12.5 and 57.5 RPM: in the Takagi-Sugeno case we only had to consider values for speeds 30 RPM and 50 RPM.

<table>
<thead>
<tr>
<th>SPEED</th>
<th>SLOW</th>
<th>MEDIUM</th>
<th>OUTPUT OF 2 RULES</th>
<th>WEIGHTED SPEED</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.5</td>
<td>0.125</td>
<td>0</td>
<td>0.125</td>
<td>1.5625</td>
</tr>
<tr>
<td>15</td>
<td>0.25</td>
<td>0</td>
<td>0.25</td>
<td>3.75</td>
</tr>
<tr>
<td>17.5</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>5.25</td>
</tr>
<tr>
<td>20</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>6</td>
</tr>
<tr>
<td>22.5</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>6.75</td>
</tr>
<tr>
<td>25</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>7.5</td>
</tr>
<tr>
<td>27.5</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>8.25</td>
</tr>
<tr>
<td>30</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>9</td>
</tr>
<tr>
<td>32.5</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>9.75</td>
</tr>
<tr>
<td>35</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>10.5</td>
</tr>
<tr>
<td>37.5</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>11.25</td>
</tr>
<tr>
<td>40</td>
<td>0.3</td>
<td>0</td>
<td>0.3</td>
<td>12</td>
</tr>
<tr>
<td>42.5</td>
<td>0.3</td>
<td>0.25</td>
<td>0.3</td>
<td>12.75</td>
</tr>
<tr>
<td>45</td>
<td>0.25</td>
<td>0.4</td>
<td>0.4</td>
<td>18</td>
</tr>
<tr>
<td>47.5</td>
<td>0.125</td>
<td>0.4</td>
<td>0.4</td>
<td>19</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>0.4</td>
<td>0.4</td>
<td>20</td>
</tr>
<tr>
<td>52.5</td>
<td>0</td>
<td>0.4</td>
<td>0.4</td>
<td>21</td>
</tr>
<tr>
<td>55</td>
<td>0</td>
<td>0.4</td>
<td>0.4</td>
<td>22</td>
</tr>
<tr>
<td>57.5</td>
<td>0</td>
<td>0.25</td>
<td>0.25</td>
<td>14.375</td>
</tr>
</tbody>
</table>

**SUM** | **5.925** | **218.6875** | **45**

The speed is **36.91 RPM**
DEFUZZIFICATION:

For Mean of Maxima for the Mamdani controller, we had to have an alpha-level cut of 0.4, and the summation ran between 45-57.5 RPM, leading to a speed of 50 RPM. We get the same result for Takagi-Sugeno controllers:

$$\eta = \frac{0.4 \times 50}{0.4} = 50 \text{ RPM}$$
**DEFUZZIFICATION:** Comparing the results of two model identification exercises – Mamdani and Takagi-Sugeno - we get the following results:

<table>
<thead>
<tr>
<th>Controller</th>
<th>Takagi-Sugeno (RPM)</th>
<th>Mamdani (RPM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centre of Area</td>
<td>41.43</td>
<td>36.91</td>
</tr>
<tr>
<td>Mean of Maxima</td>
<td>50.00</td>
<td>50.00</td>
</tr>
</tbody>
</table>
GUZZY LOGIC & GUZZY SYSTEMS

Knowledge Representation & Reasoning: The Air-conditioner Example

**DEFUZZIFICATION:** Comparing the results of two model identification exercises – Mamdani and Takagi-Sugeno - we get the following results:

Mamdani Controller and the use of COA is the best result

<table>
<thead>
<tr>
<th>Controller</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takagi-Sugeno (RPM)</td>
<td>Mamdani (RPM)</td>
</tr>
<tr>
<td>Centre of Area</td>
<td>12%</td>
</tr>
<tr>
<td>Mean of Maxima</td>
<td>35%</td>
</tr>
</tbody>
</table>
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

A formal derivation

Consider a domain where all fuzzy sets are associated with linear membership functions.

Let us denote the membership function of a fuzzy set \( A \) as \( \mu_A(x) \), \( x \in X \). All the fuzzy sets are associated with linear membership functions. Thus, a membership function is characterised by two parameters giving the greatest grade 1 and the least grade 0.

The truth value of a proposition “\( x \) is \( m_A \) and \( y \) is \( m_B \)” is expressed as

\[
\left| x \text{ is } \mu_A \text{ and } y \text{ is } \mu_B \right| = \mu_A \land \mu_B
\]
FUZZY CONTROL

FUZZY CONTROLLERS - An example

A worked example

Consider an FLC of Mamdani type:

<table>
<thead>
<tr>
<th></th>
<th>e(k)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Z</td>
</tr>
<tr>
<td>Δe(k)</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td></td>
<td>Z</td>
<td>N</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>Z</td>
</tr>
</tbody>
</table>

which expresses rules like:

Rule 1: If e(k) is negative AND Δe(k) is negative then Δu(k) is negative

Rule 9: If e(k) is positive AND Δe(k) is positive then Δu(k) is positive
FUZZY CONTROL

FUZZY CONTROLLERS- An example

A worked example

Consider an FLC of Mamdani type:

<table>
<thead>
<tr>
<th>Δe(k)</th>
<th>N</th>
<th>Z</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>N</td>
<td>N</td>
<td>Z</td>
</tr>
<tr>
<td>Z</td>
<td>N</td>
<td>Z</td>
<td>P</td>
</tr>
<tr>
<td>P</td>
<td>Z</td>
<td>P</td>
<td>P</td>
</tr>
</tbody>
</table>

The nine rules express the dependence of (change) in the value of control output on the error (the difference between expected and output values) and the change in error).

This dependence will capture some very complex non-linear, and linear relationships between \( e \) and \( Δe \) and \( Δu \).
Fuzzy Control

Fuzzy Controllers - An example

A zero-order Takagi-Sugeno Controller

<table>
<thead>
<tr>
<th>Δe(k)</th>
<th>N</th>
<th>Z</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>α₁</td>
<td>α₂</td>
<td>α₃</td>
</tr>
<tr>
<td>Z</td>
<td>α₄</td>
<td>α₅</td>
<td>α₆</td>
</tr>
<tr>
<td>P</td>
<td>α₇</td>
<td>α₈</td>
<td>α₉</td>
</tr>
</tbody>
</table>

which expresses rules like:

Rule 1: If e(k) is negative AND Δe(k) is negative then Δu(k) = α₁

Rule 9: If e(k) is positive AND Δe(k) is positive then Δu(k) = α₉
**FUZZY CONTROL**

**FUZZY CONTROLLERS - An example**

A first-order Takagi-Sugeno Controller

<table>
<thead>
<tr>
<th></th>
<th>e(k)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>∆e(k)</td>
<td>N</td>
<td>Z</td>
<td>P</td>
</tr>
<tr>
<td></td>
<td>α₁ₑ⁺β₁ ∆ₑ + δ₁</td>
<td>α₂ₑ⁺β₂ ∆ₑ + δ₂</td>
<td>α₃ₑ⁺β₃ ∆ₑ + δ₃</td>
</tr>
<tr>
<td></td>
<td>α₄ₑ⁺β₄ ∆ₑ + δ₄</td>
<td>α₅ₑ⁺β₅ ∆ₑ + δ₅</td>
<td>α₆ₑ⁺β₆ ∆ₑ + δ₆</td>
</tr>
<tr>
<td></td>
<td>α₇ₑ⁺β₇ ∆ₑ + δ₇</td>
<td>α₈ₑ⁺β₈ ∆ₑ + δ₈</td>
<td>α₉ₑ⁺β₉ ∆ₑ + δ₉</td>
</tr>
</tbody>
</table>

which expresses rules like:

Rule 1: If e(k) is negative AND ∆e(k) is negative then ∆u(k) = α₁ₑ⁺β₁ ∆ₑ + δ₁

Rule 9: If e(k) is positive AND ∆e(k) is positive then ∆u(k) = α₉ₑ⁺β₉ ∆ₑ + δ₉
Takagi and Sugeno have described a fuzzy implication $R$ is of the format:

$$R: \text{if} \ (x_1 \text{ is } \mu_A(x_1), \ldots, x_k \text{ is } \mu_A(x_k)) \ \text{then} \ y = g(x_1, \ldots, x_k), \text{ where:}$$

<table>
<thead>
<tr>
<th>$y$</th>
<th>Variable of the consequence whose value is inferred</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1, \ldots, x_k$</td>
<td>Variables of the premise that appear also in the part of the consequence</td>
</tr>
<tr>
<td>$m_A(x_1),\ldots, m_A(x_1)$</td>
<td>Fuzzy sets with linear membership functions representing a fuzzy subspace in which the implication $R$ can be applied for reasoning.</td>
</tr>
<tr>
<td>$f$</td>
<td>Logical function connects the propositions in the premise.</td>
</tr>
<tr>
<td>$g$</td>
<td>Function that implies the value of $y$ when $x_1,\ldots, x_k$ satisfies the premise.</td>
</tr>
</tbody>
</table>
In the premise if $\mu_A(x_i)$ is equal to $X_i$ for some $i$ where $X_i$ is the universe of discourse of $x_i$, this term is omitted; $x_i$ is unconditioned; otherwise $x_i$ is regarded as conditioned.
In the premise if $\mu_A(x_i)$ is equal to $X_i$ for some $i$ where $X_i$ is the universe of discourse of $x_i$, this term is omitted; $x_i$ is unconditioned. The following example will help in clarifying the argumentation related to 'conditioned' and 'unconditioned' terms in a given implication:

\[ R: \text{ if } x_1 \text{ is small and } x_2 \text{ is big then } y = x_1 + x_2 + 2x_3. \]

The above implication comprises two conditioned premises, $x_1$ and $x_2$, and one unconditioned premise, $x_3$.

The implication suggests that if $x_1$ is small and $x_2$ is big, then the value of $y$ would depend upon and be equal to the sum of $x_1$, $x_2$, and $2x_3$, where $x_3$ is unconditioned in the premise.
Typically, for a Takagi-Sugeno controller, an implication is written as:

\[
R: \text{if } x_1 \text{ is } \mu_1 \text{ and } \ldots \text{ and } x_k \text{ is } \mu_k
\]
\[
\text{then } y = p_0 + p_1 x_1 + \ldots + p_k x_k.
\]

The assumption here is that only ‘and’ connectives are used in the antecedants or premises of the rules. And, that the relationship between the output and inputs is strictly a LINEAR (weighted average) relationship. (The weights here are \(p_0, p_1, \ldots, p_k\).)
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

Reasoning Algorithm
Recall the arguments related to the membership functions of the union and intersection of fuzzy sets. The intersection of sets $A$ and $B$ is given as:

$$\mu_{A \cap B} = \min (\mu_A, \mu_B)$$

We started this discussion by noting that we will explore the problems of multivariable control (**MultipleInputSingleOutput**). Usually, the rule base in a fuzzy control system comprises a number of rules; in the case of multivariable control the relevant rules have to be tested for what they imply.
Consider a system with \( n \) implications (rules); the variable of consequence, \( y \), will have to be notated for each of these implications, leading to \( y_i \) variables of consequence. There are three stages of computations in Takagi-Sugeno controllers:

**FUZZIFICATION**: Fuzzify the input. For all input variables compute the implication for each of the rules;

**INFERENC**E or **CONSEQUENCES**: For each implication compute the consequence for a rule which fires. Compute the output \( y \) for the rule by using the linear relationship between the inputs and the output \( (y = p_0 + p_1x_1 + \ldots + p_kx_k) \).

**AGGREGATE (& DEFUZZIFICATION)**: The final output \( y \) is inferred from \( n \)-implications and given as an average of all individual implications \( y_i \) with weights \( |y = y_i| \):

\[
y = \frac{\sum |y = y_i| \times y_i}{\sum |y = y_i|}
\]

where \( |y = y_i| \) stands for the truth value of a given proposition.
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

Consider the following fuzzy implications (or rules) \( R_1, R_2, R_3 \) used in the design of a Takagi-Sugeno controller:

\[
\begin{align*}
R_1 & \rightarrow \text{If } x_1 \text{ is } \text{small}_1 \text{ & } x_2 \text{ is } \text{small}_2 \quad \text{then } y^{(1)} = x_1 + x_2 \\
R_2 & \rightarrow \text{If } x_1 \text{ is } \text{big}_1 \quad \text{then } y^{(2)} = 2x_1 \\
R_3 & \rightarrow \text{If } x_2 \text{ is } \text{big}_2 \quad \text{then } y^{(3)} = 3x_2
\end{align*}
\]

where \( y^{(i)} \) refers to the consequent variable for each rule labelled \( R_i \) and \( x_1 \) and \( x_2 \) refer to the input variables that appear in premise of the rules.
**FUZZY CONTROL**

**FUZZY CONTROLLERS – Takagi-Sugeno Controllers**

The membership function for $small_1$, $small_2$, $big_1$ and $big_2$ are given as follows

<table>
<thead>
<tr>
<th>$x$</th>
<th>Small1</th>
<th>Small2</th>
<th>Big1</th>
<th>Big2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.938</td>
<td>0.875</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.875</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.813</td>
<td>0.625</td>
<td>0</td>
<td>0.125</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>0.5</td>
<td>0</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>0.688</td>
<td>0.375</td>
<td>0</td>
<td>0.375</td>
</tr>
<tr>
<td>6</td>
<td>0.625</td>
<td>0.25</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>7</td>
<td>0.563</td>
<td>0.125</td>
<td>0</td>
<td>0.625</td>
</tr>
<tr>
<td>8</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0.75</td>
</tr>
<tr>
<td>9</td>
<td>0.438</td>
<td>0</td>
<td>0</td>
<td>0.875</td>
</tr>
<tr>
<td>10</td>
<td>0.375</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>0.313</td>
<td>0</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
<td>0</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>0.188</td>
<td>0</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>14</td>
<td>0.125</td>
<td>0</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>0.063</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>0</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>0</td>
<td>0.7</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>0</td>
<td>0</td>
<td>0.9</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
The membership function for $small_1$, $small_2$, $big_1$ and $big_2$ are given as follows.

![Membership Function Diagram](image-url)
Fuzzy Control

Fuzzy Controllers – Takagi-Sugeno Controllers An example

Let us compute the FINAL OUTPUT $y$ for the following values:

$x_1 = 12 \& x_2 = 5$

using Takagi and Sugeno’s formula:

$$y = \frac{\sum |y = y^i| \times y^i}{\sum |y = y^i|}$$

where $|y = y^i|$ stands for the truth value of a given proposition.
**FUZZY CONTROL**

**FUZZY CONTROLLERS – Takagi-Sugeno Controllers**  
**An example**

**FUZZUFACTION:** We have the following values of the membership functions for the two values \( x_1 = 12 \) & \( x_2 = 5 \):

<table>
<thead>
<tr>
<th></th>
<th>Small1</th>
<th>Small2</th>
<th>Big1</th>
<th>Big2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 = 12 )</td>
<td>12</td>
<td>0.25</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>( x_2 = 5 )</td>
<td>0.6875</td>
<td>0.375</td>
<td>0</td>
<td>0.375</td>
</tr>
</tbody>
</table>
**Fuzzy Control**

**Fuzzy Controllers** – Takagi-Sugeno Controllers: An example

**Inference & Consequence:**

$x_1 = 12 \& x_2 = 5$

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premise 1</th>
<th>Premise 2</th>
<th>Consequent ( y )</th>
<th>Truth Value ( \text{Min}(\text{Premise 1} &amp; \text{Premise 2}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1 )</td>
<td>\text{Small}_1(x_1) = 0.25</td>
<td>\text{Small}_2(x_2) = 0.375</td>
<td>( y^{(1)} = x_1 + x_2 ) ( = 12 + 5 )</td>
<td>\text{Min}(0.25, 0.375) = 0.25</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>\text{Big}_1(x_1) = 0.2</td>
<td></td>
<td>( y^{(2)} = 2x_1 ) ( = 24 )</td>
<td>0.2</td>
</tr>
<tr>
<td>( R_3 )</td>
<td>\text{Big}_2(x_2) = 0.375</td>
<td></td>
<td>( y^{(3)} = 3x_2 ) ( = 15 )</td>
<td>0.375</td>
</tr>
</tbody>
</table>
Fuzzy Control

Fuzzy Controllers – Takagi-Sugeno Controllers

An example

**AGGREGATION (&DEFUZZIFICATION):**

\[
x_1 = 12 \& x_2 = 5
\]

\[
y = \frac{\sum |y = y^i| \cdot y^i}{\sum |y = y^i|}
\]

Using a Centre of Area computation for \( y \) we get:

\[
y = \frac{\sum_{i=1,3} |y = y^{(i)}| \cdot y^{(i)}}{\sum_{i=1,3} |y = y^{(i)}|}
\]

\[
y = \frac{0.25 \cdot 17 + 0.2 \cdot 24 + 0.375 \cdot 15}{0.25 + 0.2 + 0.375} = 17.8
\]
Key difference between a Mamdani-type fuzzy system and the Takagi-Sugeno-Kang System?

A zero-order Sugeno fuzzy model can be viewed as a special case of the Mamdani fuzzy inference system in which each rule is specified by fuzzy singleton or a pre-defuzzified consequent. In Sugeno’s model, each rule has a crisp output, the overall input is obtained by a weighted average – this avoids the time-consuming process of defuzzification required in a Mandani model. The weighted average operator is replaced by a weighted sum to reduce computation further. (Jang, Sun, Mizutani (1997:82)).
FUZZY CONTROL

FUZZY CONTROLLERS – Takagi-Sugeno Controllers

Takagi and Sugeno (1985) have argued that in order to develop a generic and simple mathematical tool for computing fuzzy implications one needs to look at a fuzzy partition of fuzzy input space. In each fuzzy subspace a linear input-output relation is formed. The output of fuzzy reasoning is given by the values inferred by some implications that were applied to an input.
Knowledge Representation & Reasoning: The Air-conditioner Example

Takagi-Sugeno model is an approximation of a Mamdani controller – in that the Takagi-Sugeno model ignores the fuzziness of linguistic variables in the consequent, but accounts for the fuzziness of variables in the antecedents; whereas Mamdani controller takes into the fuzziness of variables appearing both in the antecedents and the consequent.
Model Identification: Given a choice between two models, say Mamdani and Takagi-Sugeno, we have to first identify why to choose a fuzzy logic system (will a crisp description not work as it is simpler to compute) and second whether to use an elaborate model (say Mamdani) rather than an approximation to the model (say, Takagi-Sugeno).

The choice can be based on the relative performance of the two models.