Fuzzy Logic and Fuzzy Systems – Properties & Relationships

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https://www.cs.tcd.ie/Khurshid.Ahmad/Teaching/Teaching.html 1

FUZZY LOGIC & FUZZY SYSTEMS

Terminology

- Fuzzy sets are sets whose elements have degrees of membership of the sets.
- Fuzzy sets are an extension of the classical set.
- Membership of a set governed by classical set theory is described according to a bivalent condition — all members of the set definitely belong to the set whilst all non-members do not belong to the classical set.
- Sets governed by the rules of classical set theory are referred to as **crisp sets**.

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FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

The concept of a set and set theory are powerful concepts in mathematics. However, the principal notion underlying set theory, that an element can (exclusively) either belong to set or not belong to a set, makes it well nigh impossible to represent much of human discourse. How is one to represent notions like:

***large* profit**

***high* pressure**

***tall* man**

***wealthy* woman**

***moderate* temperature.**

Ordinary set-theoretic representations will require the maintenance of a crisp differentiation in a very artificial manner:

high, high to some extent, not quite high, very high etc.

FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

'Many decision-making and problem-solving tasks are too complex to be understood quantitatively, however, people succeed by using knowledge that is imprecise rather than precise. Fuzzy set theory, originally introduced by Lotfi Zadeh in the 1960's, resembles human reasoning in its use of approximate information and uncertainty to generate decisions. It was specifically designed to mathematically represent uncertainty and vagueness and provide formalized tools for dealing with the imprecision intrinsic to many problems. By contrast, traditional computing demands precision down to each bit. Since knowledge can be expressed in a more natural by using fuzzy sets, many engineering and decision problems can be greatly simplified.' ***http://www.emsl.pnl.gov:2080/proj/neuron/fuzzy/what.html***

FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

Lotfi Zadeh introduced the *theory of fuzzy sets*: A fuzzy set is a collection of objects that might belong to the set to a degree, varying from 1 for full belongingness to 0 for full non-belongingness, through all intermediate values

Zadeh employed the concept of a membership function assigning to each element a number from the unit interval to indicate the intensity of belongingness. Zadeh further defined basic operations on fuzzy sets as essentially extensions of their conventional ('ordinary') counterparts.

Lotdfi Zadeh, Professor in the Graduate School, Computer Science Division Department of Elec. Eng. and Comp Sciences, University of California Berkeley, CA 94720 -1776 Director, Berkeley Initiative in Soft Computing (BISC)

http://www.cs.berkeley.edu/People/Faculty/Homepages/zadeh.html

In 1995, Dr. Zadeh was awarded the IEEE Medal of Honor "For pioneering development of fuzzy logic and its many diverse applications." In 2001, he received the American Computer Machinery's 2000 Allen Newell Award for seminal contributions to AI through his development of fuzzy logic.

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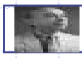
FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

‘A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one.’ (Zadeh 1965:338)

‘The notions of inclusion, union, intersection, complement, relation, convexity, [..] can be extended to such sets, and various properties of these notions in the context of fuzzy sets [..] [have been] established.’ (*ibid*).



Lotfi Zadeh (1965). ‘Fuzzy Sets’. *Information and Control*. Volume 8, Issue 3, June 1965, Pages 338-353

FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

'The notion of an event and its probability constitute the most basic concepts of probability theory. […] An event [typically] is a precisely specified collection of points in the sample space.' (Zadeh 1968:421).

Consider some everyday events and occurrences:

It is a *cold* day;

My computer is *approximately* 5KG in weight;

In 20 tosses of a coin there are *several* more heads than tails

In everyday contexts an event 'is a fuzzy rather than a sharply defined collection of points'. Using the concept of a fuzzy set, 'the notions of an event and its probability can be extended in a natural fashion to fuzzy events of the type [described above]' (*ibid*)

Lotfi Zadeh (1968). 'Probability Measures of Fuzzy Sets'. *J. Mathematical Analysis & Applications* Volume 23, Issue 2, August 1968, Pages 421-427

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FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

'In sharp contrast to the idealized world of mathematics, our perception of the real world is pervaded by concepts which do not have sharply defined boundaries, e.g., *tall, fat, many, most, slowly, old. familiar, relevant, much larger than, kind,* etc. A key assumption in fuzzy logic is that the denotations of such concepts are *fuzzy sets,* that is, classes of objects in which the transition from membership to non-membership is gradual rather than abrupt.' (Zadeh 1990:99).

Lotfi Zadeh (1990). 'Probability Measures of Fuzzy Sets'. *International Journal of General Systems*. Vol. 17, pp. 95-105

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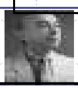
FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

System	Variable	Relationships	
		Simple	Complex
Conventional	**Quantitative, e.g. numerical**	**Conditional and Relational Statements between domain objects *A, B*:** **IF *A* THEN B;** ***A* is-a-part-of *B*** ***A* weighs 5KG**	**Ordered sequences of instructions comprising** ***A*=5;** **IF *A* < *5* THEN** ***B=A+5*** **………**
Fuzzy	**Quantitative (e.g. numerical) and *linguistic* variables**	**Conditional and Relational Statements between domain objects *A, B*:** **IF *A* (Ψ_A) THEN *B* (Ψ_B)** ***A* weighs *about* 5KG**	**Ordered sequences of instructions comprising** ***A* IS-SMALL;** **IF *A* IS_SMALL THEN *B* IS_LARGE**



Lotfi Zadeh (1990). 'Probability Measures of Fuzzy Sets'. *International Journal of General Systems*. Vol. 17, pp. 95-105

FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

A FUZZY SYSTEM can be contrasted with a CONVENTIONAL (CRISP) System in three main ways:

1. **A linguistic variable is defined as a variable whose values are sentences in a natural or artificial language. Thus, if *tall, not tall, very tall, very very tall*, etc. are values of *HEIGHT*, then *HEIGHT* is a linguistic variable.**
2. **Fuzzy conditional statements are expressions of the form *IF A THEN B*, where *A* and *B* have fuzzy meaning, e.g., *IF x is small THEN y is large*, where small and large are viewed as labels of fuzzy sets.**
3. **A fuzzy algorithm is an ordered sequence of instructions which may contain fuzzy assignment and conditional statements, e.g., *x =very small, IF x is small THEN y is large*. The execution of such instructions is governed by the compositional rule of inference and the rule of the preponderant alternative.**

Lotfi Zadeh (1990). 'Probability Measures of Fuzzy Sets'. *International Journal of General Systems*. Vol. 17, pp. 95-105

FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

Zadeh also devised the so-called fuzzy logic: This logic was devised model 'human' reasoning processes comprising:

vague predicates:	**e.g. *large, beautiful, small***
partial truths:	**e.g. *not very* true, *more* or *less* false**
linguistic quantifiers:	**e.g. *most, almost all, a few***
linguistic hedges:	**e.g. *very, more* or *less.***

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FUZZY LOGIC & FUZZY SYSTEMS

BACKGROUND & DEFINITIONS

The notion of fuzzy restriction is crucial for the fuzzy set theory:

A FUZZY RELATION ACTS AS AN ELASTIC CONSTRAINT ON THE VALUES THAT MAY BE ASSIGNED TO A VARIABLE.

Calculus of Fuzzy Restrictions is essentially a body of concepts and techniques for dealing with fuzzy restrictions in a systematic way: to furnish a conceptual basis for approximate reasoning - neither exact nor inexact reasoning.(cf. Calculus of Probabilities and Probability Theory)

FUZZY LOGIC & FUZZY SYSTEMS

UNCERTAINITY AND ITS TREATMENT

Theory of fuzzy sets and fuzzy logic has been applied to problems in a variety of fields:

Taxonomy; Topology; Linguistics; Logic; Automata Theory; Game Theory; Pattern Recognition; Medicine; Law; Decision Support; Information Retrieval;

And more recently FUZZY Machines have been developed including automatic train control and tunnel digging machinery to washing machines, rice cookers, vacuum cleaners and air conditioners.

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FUZZY LOGIC & FUZZY SYSTEMS

UNCERTAINITY AND ITS TREATMENT

Fuzzy set theory has a number of branches:

Fuzzy mathematical programming

(Fuzzy) Pattern Recognition

(Fuzzy) Decision Analysis

Fuzzy Arithmetic

Fuzzy Topology

&

Fuzzy Logic

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FUZZY LOGIC & FUZZY SYSTEMS

UNCERTAINITY AND ITS TREATMENT

The term fuzzy logic is used in two senses:

•**Narrow sense**: Fuzzy logic is a branch of fuzzy set theory, which deals (as logical systems do) with the representation and inference from knowledge. Fuzzy logic, unlike other logical systems, deals with ***imprecise*** or ***uncertain*** knowledge. In this narrow, and perhaps correct sense, fuzzy logic is just one of the branches of fuzzy set theory.

•Broad Sense: fuzzy logic synonymously with fuzzy set theory

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FUZZY LOGIC & FUZZY SYSTEMS

Properties

Elements of a fuzzy set may belong to the set, may not belong to the set, or may belong to a **degree**.

Membership of a crisp set is described by a bivalent condition; the membership of a fuzzy set is described by a multi-valent condition.

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS

An Example: Consider a set of numbers: X = {1, 2, ….. 10}. Johnny's understanding of numbers is limited to 10, when asked he suggested the following. Sitting next to Johnny was a fuzzy logician noting :

'Large Number'	Comment	'Degree of membership'
10	'Surely'	1
9	'Surely'	1
8	'Quite poss.'	0.8
7	'Maybe'	0.5
6	'In some cases, not usually'	0.2
5, 4, 3, 2, 1	'Definitely Not'	0

FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS

<u>An Example</u>: Consider a set of numbers: X = {1, 2, ….. 10}. Johnny's understanding of numbers is limited to 10, when asked he suggested the following. Sitting next to Johnny was a fuzzy logician noting :

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6	'In some cases, not usually'	0.2
5, 4, 3, 2, 1	'Definitely Not'	0

We can denote Johnny's notion of 'large number' by the fuzzy set

$$A = 0/1 + 0/2 + 0/3 + 0/4 + 0/5 + 0.2/6 + 0.5/7 + 0.8/8 + 1/9 + 1/10$$

FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS

Fuzzy (sub-)sets: Membership Functions

For the sake of convenience, usually a fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

that belongs to a finite universe of discourse:

$$\underset{\sim}{A} \subset \{x_1, x_2, \ldots\ldots\ldots, \quad x_n\} \equiv X$$

where $\mu_A(x_i)/x_i$ (a singleton) is a pair "grade of membership element".

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS

Johnny's large number set membership function can be denoted as:

'Large Number'	μ (.)
10	$\mu_A (10) = 1$
9	$\mu_A (9) = 1$
8	$\mu_A (8) = 0.8$
7	$\mu_A (7) = 0.5$
6	$\mu_A (6) = 0.2$
5, 4, 3, 2, 1	$\mu_A (5) = \mu_A (4) = \mu_A (3) = \mu_A (2) = \mu_A (10) = 0$

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FUZZY SETS

Johnny's large number set membership function can be used to define 'small number' set *B*, where

$$\mu_B\,(.) = NOT\,(\mu_A\,(.)\,) = 1 - \mu_A\,(.):$$

'Small Number'	μ (.)
10	$\mu_B(10) = 0$
9	$\mu_B\,(9) = 0$
8	$\mu_B\,(8) = 0.2$
7	$\mu_B\,(7) = 0.5$
6	$\mu_B\,(6) = 0.8$
5, 4, 3, 2, 1	$\mu_B\,(5) = \mu_B\,(4) = \mu_B\,(3) = \mu_B\,(2) = \mu_B\,(1) = 1$

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FUZZY SETS

Johnny's large number set membership function can be used to define 'very large number' set *C*, where

$$\mu_C\,(.) = DIL(\mu_A\,(.)\,) = \mu_A\,(.) * \mu_A\,(.)$$

and 'largish number' set *D*, where

$$\mu_D\,(.) = CON(\mu_A\,(.)\,) = SQRT(\mu_A\,(.))$$

Number	Very Large (μ_C (.))	Largish (μ_D (.))
10	1	1
9	1	1
8	0.64	0.89
7	0.25	0.707
6	0.04	0.447
5, 4, 3, 2, 1	0	0

FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS

Fuzzy (sub-)sets: Membership Functions

Let X = { x } be a universe of discourse i.e., a set of all possible, e.g., feasible or relevant, elements with regard to a fuzzy (vague) concept (property). Then

$$A \underset{\sim}{\subset} X \ (A \text{ of } X)$$

denotes a fuzzy subset, or loosely fuzzy set, a set of ordered pairs $\{(x, \mu_A(x))\}$ where $X \in x$.

$\mu_A : X \rightarrow [0, 1]$ the membership function of A

$\mu_A(x) \in [0, 1]$ is grade of membership of x in A

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FUZZY SETS

Fuzzy (sub-)sets: Membership Functions

$$\mu_A(x) \equiv A(x)$$

- Many authors denote the membership grade $\mu_A(x)$ by $A(x)$.
- A FUZZY SET IS OFTEN DENOTED BY ITS MEMBERSHIP FUNCTION
- If [0, 1] is replaced by {0, 1}:This definition coincides with the characteristic function based on the definition of an ordinary, i.e., non-fuzzy set.

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FUZZY SETS

Like their ordinary counterparts, fuzzy sets have well defined properties and there are a set of operations that can be performed on the fuzzy sets. These properties and operations are the basis on which the fuzzy sets are used to deal with uncertainty on the hand and to represent knowledge on the other.

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS: PROPERTIES

Properties	Definition
P_1	Equality of two fuzzy sets
P_2	Inclusion of one set into another fuzzy set
P_3	Cardinality of a fuzzy set
P_4	An empty fuzzy set
P_5	α-cuts

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS: PROPERTIES

Properties	Definition	Examples
P_1	Fuzzy set A is considered equal to a fuzzy set B, IF AND ONLY IF (*iff*) $\mu_A(x) = \mu_B(x)$	
P_2	Inclusion of one set into another fuzzy set $A \subset X$ is included in (is a subset of) another fuzzy set, $B \subset X$ $\mu_A(x) \leq \mu_B(x)\ \forall x \in X$	Consider $X = \{1, 2, 3\}$ and $A = 0.3/1 + 0.5/2 + 1/3$; $B = 0.5/1 + 0.55/2 + 1/3$ Then A is a subset of B

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS: PROPERTIES

Properties	Definition
P_3	Cardinality of a non-fuzzy set, Z, is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $CARD_A = \mu_A(x_1) + \mu_A(x_2) + \ldots + \mu_A(x_n) = \sum_{i=1}^{n} \mu_A(x_i)$ Example: Card $_A$ = 1.8 Card $_B$ = 2.05

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FUZZY SETS: PROPERTIES

Properties	Definition	Examples
P_4	A fuzzy set A is empty, IF AND ONLY IF $\mu_A(x) = 0$, $\forall x \in X$	
P_5	An α-cut or α-level set of a fuzzy set $A \subset X$ is an ORDINARY SET $A_\alpha \subset X$, such that $A_\alpha = \{x \in X;\ \mu_A(x) \geq \alpha\}$. Decomposition $A = \sum_{0 \leq \alpha \leq 1} \alpha A_\alpha$	$A = 0.3/1 + 0.5/2 + 1/3 \rightarrow X = \{1, 2, 3\}$ $A_{0.5} = \{2, 3\}$, $A_{0.1} = \{1, 2, 3\}$, $A_1 = \{3\}$

FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS: OPERATIONS

Operations	Definition
O_1	Complementation
O_2	Intersection
O_3	Union
O_4	Bounded sum
O_5	Bounded difference
O_6	Concentration
O_7	Dilation

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FUZZY LOGIC & FUZZY SYSTEMS

FUZZY SETS: OPERATIONS

Operations	Definition & Example
O_1	The complementation of a fuzzy set $A \subset X$ (A of X) $\rightarrow \urcorner A$ (*NOT A* of *X)* $\sim$ $\Rightarrow \mu_{\urcorner A}(x) = 1 - \mu_A(x)$
	Example: Recall X = {1, 2, 3} and A = 0.3/1 + 0.5/2 + 1/3 $\rightarrow$A' = $\urcorner$A = 0.7/1 + 0.5/2.
	Example: Consider Y = {1, 2, 3, 4} and $C \subset Y \Rightarrow$ $\sim$ C = 0.6/1 + 0.8/2 + 1/3; then C' = ($\urcorner$C) = 0.4/1 + 0.2/2 + <u>1/4</u> C' contains one member not in C (i.e., *4*) and does not contain one member of C (i.e., *3*)

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Properties

More formally,

Let *X* be some universe of discourse
Let *S* be a subset of *X*

Then, we define a 'characteristic function' or 'membership function' μ.

Properties

The membership function associated with S is a mapping

$$\mu_s : X \rightarrow \{0,1\}$$

Such that for any element $x \varepsilon X$

If $\mu_S(x)=1$, then x is a member of the set S,
If $\mu_S(x)=0$, then x is a not a member of the set S

Properties

Remember the curly brackets ({ and }) are used to refer to binary value

$$\mu_S : X \rightarrow \{0,1\}$$

For fuzzy subset (A) we use square brackets ([and]) to indicate the existence of a UNIT INTERVAL

$$\mu_A : X \rightarrow [0,1]$$

Properties

For each x in the universe of discourse X, the function μ_A is associated with the fuzzy subset A.

$$\mu_A : X \to [0,1]$$

$\mu_A(x)$ indicates to the degree to which x belongs to the fuzzy subset A.

FUZZY LOGIC & FUZZY SYSTEMS

Properties

A fuzzy subset of X is called normal if there exists at least one element $\chi \in X$ such that $\mu_A(\chi)=1$.

A fuzzy subset that is not normal is called subnormal.

$\Rightarrow$All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.

The height of a fuzzy subset A is the largest membership grade of an element in A

$$height(A) = \max_{\chi}(\mu_A(\chi))$$

FUZZY LOGIC & FUZZY SYSTEMS

Properties

- Assume A is a fuzzy subset of X; the <u>support</u> of A is the crisp subset of X whose elements all have non-zero membership grades in A:

$$supp(\mathrm{A}) = \{x \mid \mu_{\mathrm{A}}(x) > 0 \text{ and } x \in \mathrm{X}\}$$

Assume A is a fuzzy subset of X; the <u>core</u> of A is the crisp subset of X consisting of all elements with membership grade 1:

$$\mathrm{Core(A)} = \{x \mid \mu_{\mathrm{A}}(x) = 1 \text{ and } x \in \mathrm{X}\}$$

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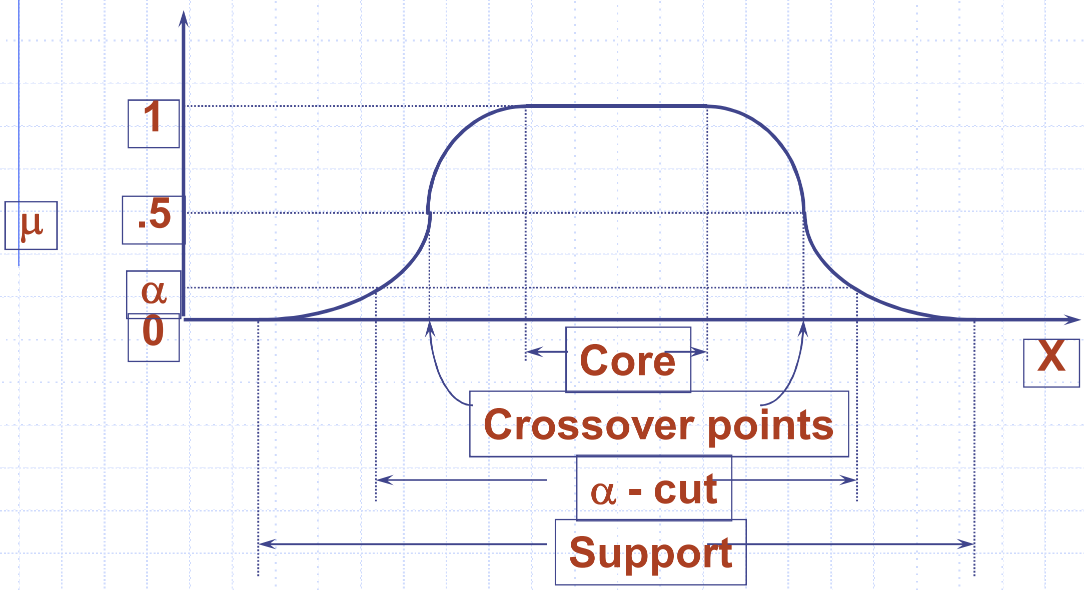
FUZZY LOGIC & FUZZY SYSTEMS
Summary



FUZZY LOGIC & FUZZY SYSTEMS

Properties

A normal fuzzy subset has a non-null core while a subnormal fuzzy subset has a null core.

Example:

Consider two fuzzy subsets of the set X,

X = {a, b, c, d, e }

referred to as A and B

A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

and

B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

FUZZY LOGIC & FUZZY SYSTEMS

Properties

From the properties above we have:

•Normal/Subnormal?

A ≡> Normal fuzzy set (element *a* has unit membership)

B ≡> Subnormal fuzzy set (no element has unit membership)

•Height:

height (A) = 1 $\Longrightarrow \max_{k = a} \{\mu_A(k)\}$

height (B) = 0.9 $\Longrightarrow \max_{k = a} \{\mu_A(k)\}$

FUZZY LOGIC & FUZZY SYSTEMS

Properties

Support:

Supp(A) = {a, b, c, d } (e has zero membership)

Supp(B) = {a, b, c, d, e }

FUZZY LOGIC & FUZZY SYSTEMS

Properties

Core:

Core(A) = {a} (only unit membership)

Core(B) = Ø (no element with unit membership)

Cardinality:

$$\text{Card}(A) = \sum_{k=a}^{e} \mu_A(k) = 1 + 0.3 + 0.2 + 0.8 + 0 = 2.3$$

Card(B) = 0.9+0.6+0.1+0.3+0.2=2.1

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

Operations:
The union of fuzzy subsets, A and B, of the set X, is denoted as the fuzzy subset C of X.

$$\boldsymbol{C = A \cup B} \textbf{ such that for each } \chi \in \mathbf{X}$$
$$\mu_C(x) = \max[\mu_A(x), \mu_B(x)] = \mu_A(x) \vee \mu_B(x)$$

The intersection of the fuzzy subsets A and B is denoted as the fuzzy subset D of X

$$\boldsymbol{D = A \cap B} \textbf{ for each } x \in \mathbf{X}$$
$$\mu_D(x) = \min[(\mu_A(x), \mu_B(x)]$$

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

The operations of *Max* and *Min* play a fundamental role in fuzzy set theory and are usually computed from the following formulae:

$$Max(a, b) = \frac{a + b + |a - b|}{2}$$

$$Min(a, b) = \frac{a + b - |a - b|}{2}$$

FUZZY LOGIC & FUZZY SYSTEMS

Operations

Example: Union and Intersection of Fuzzy sets

Recall

A = {1/a, 0.3/b, 0.2/c, 0.8/d, 0/e}

B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

The *union* of A and B is

$\boldsymbol{C = A \cup B}$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e},

maximum of the membership functions for A and B

and the *intersection* of A and B is

$\boldsymbol{D = A \cap B}$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

minimum of the membership functions for A and B

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

The **complement** or **negation** of a fuzzy subset A of X is denoted by

$$\overline{A} = X - A$$

and the membership function of the **complement** is given as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

THE NEGATION IS THE COMPLEMENT OF A WITH RESPECT TO THE WHOLE SPACE X.

EXAMPLE:

$$A = \{1/a,\quad 0.3/b,\quad 0.2/c,\quad 0.8/d,\quad 0/e\}$$

$$\overline{A} = \{0/a,\quad 0.7/b,\quad 0.8/c,\quad 0.2/d,\quad 1/e\}$$

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

Generally, the intersection of a fuzzy subset and its complement is NOT the NULL SET.

$$E = \overline{A} \cap A \neq \phi;$$

EXAMPLE:

$$\mu_E(x) = Min\ [(\mu_{\overline{A}}(x), \mu_A(x)]$$

$$\therefore E = \{0/a, 0.3/b, 0.2/c, 0.2/d, 0/e\}.$$

The distinction between a fuzzy set and its complement, especially when compared with the distinction between a crisp set and its complement, is not as clear cut. The above example shows that fuzzy subset *E*, the intersection of *A* and its complement, still has three members.

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

If A is a fuzzy subset of X and α is any non-negative number, then A^{α} is the fuzzy subset B such that:

$$\mu_B(x) = (\mu_A(x))^{\alpha}$$

EXAMPLE:

$$A = \{1/a, 0.6/b, 0.3/c, 0/d, 0.5/e\}$$

$$A^2 = \{1/a, 0.36/b, 0.09/c, 0/d, 0.25/e\}$$

$$\sqrt{A} = A^{1/2} = \{1/a, 0.774/b, 0.548/c, 0/d, 0.707/e\}$$

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

CONCENTRATION:

If $\alpha > 1$ then $A^{\alpha} \subset A$ ➔ decreases membership

DILATION

If $\alpha < 1$ then $A^{\alpha} \supset A$ ➔ increases membership.

Note: If A is a crisp subset and $\alpha > 0$, then $A^{\alpha} = A$

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

Level Set

If A is a fuzzy subset of X and $0 \le \alpha \le 1$
Then we can define another fuzzy subset F such that

$$F = \alpha A; \qquad \mu_F(x) = \alpha\, \mu_A(x) \quad x \,\varepsilon\, X$$

EXAMPLE:
Let $\alpha = 0.5$, and

A = {1/a, 0.3/b, 0.2/c, 0.8/d, 0/e}

Then

F = {0.5/a, 0.15/b, 0.1/c, 0.4/d, 0/e}

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FUZZY LOGIC & FUZZY SYSTEMS

Operations

Level Set

The α-level set of the fuzzy subset *A* (of *X*) is the CRISP subset of *X* consisting of all the elements in *X*, such that:

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha, \quad x \,\varepsilon\, X\}$$

EXAMPLE:

$$A = \{1/a, 0.3/b, 0.2/c, 0.8/d, 0/e\}$$

α − level subsets :

$$A_\alpha = \{a, b, c, d\}, \qquad 0 < \alpha \leq 0.2;$$

$$A_\alpha = \{a, b, d\}, \qquad 0.2 < \alpha \leq 0.3;$$

$$A_\alpha = \{a, d\}, \qquad 0.3 < \alpha \leq 0.8;$$

$$A_\alpha = \{a\}, \qquad 0.8 < \alpha \leq 1.$$

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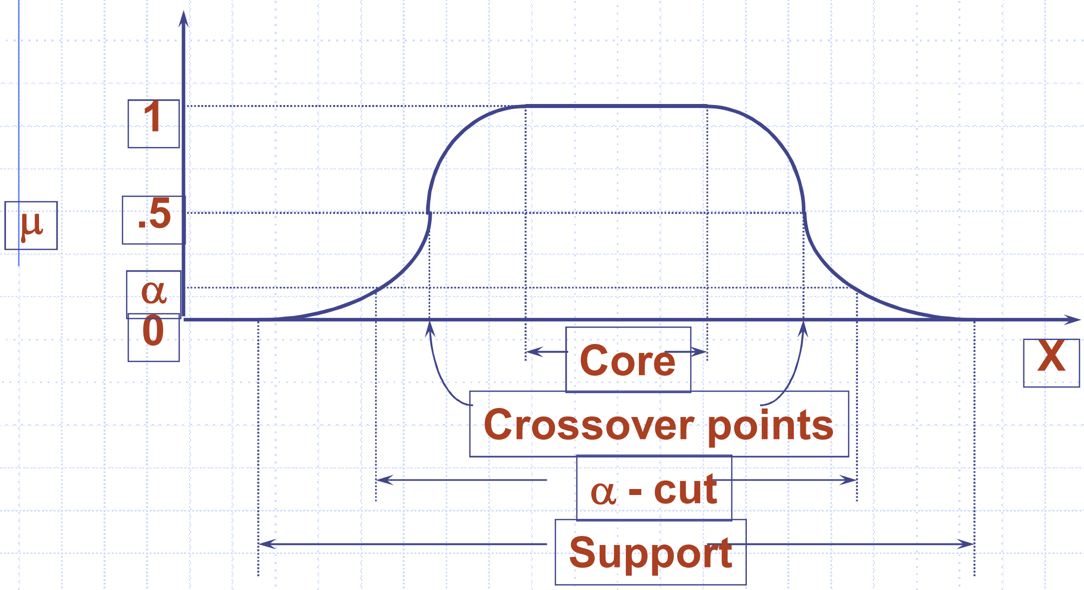
FUZZY LOGIC & FUZZY SYSTEMS

Summary



FUZZY LOGIC & FUZZY SYSTEMS

Membership Functions

- Triangular MF: $$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF: $$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Gaussian MF: $$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF: $$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

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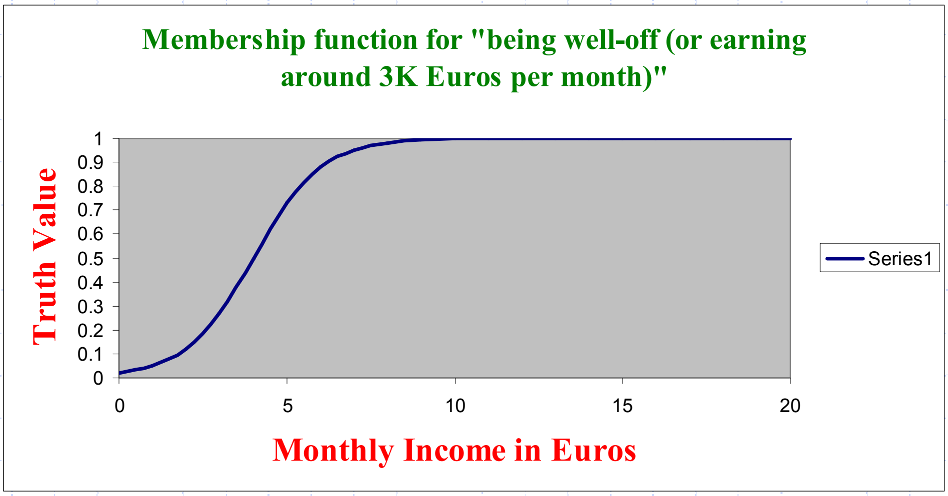
FUZZY LOGIC & FUZZY SYSTEMS

Membership Functions: Sigmoid Function

$$\mu_{wealthy}(x) = \frac{1}{1+e^{(-a \times (x-c))}}$$



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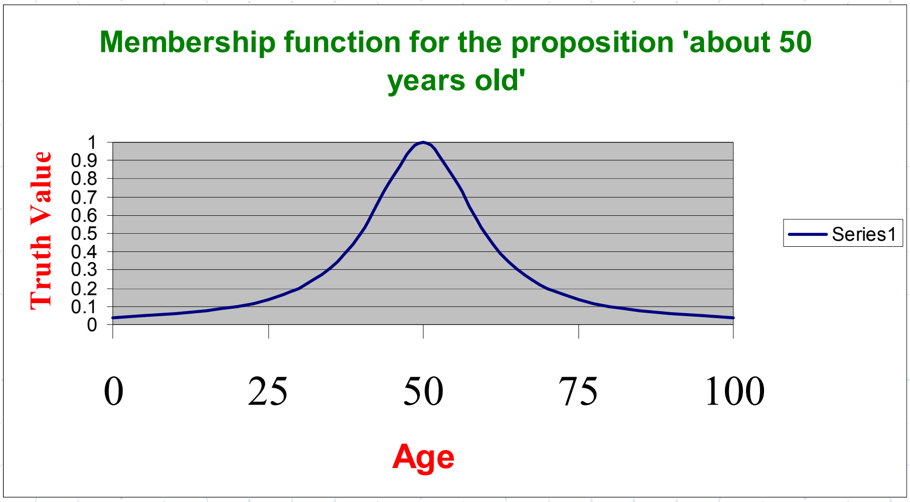
FUZZY LOGIC & FUZZY SYSTEMS

Membership Functions



$$\mu_{50\ or\ so\ old}(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^2}$$

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

The ***cartesian*** or ***cross product*** of fuzzy subsets *A* and *B*, of sets *X* and *Y* respectively is denoted as

$$\boldsymbol{A \times B}$$

This cross product relationship *T* on the set $\boldsymbol{X \times Y}$ is denoted as

$$\boldsymbol{T = A \times B}$$

$$\mu_T(x, y) = MIN\ [(\mu_A(x),\ \mu_B(y))]$$

EXAMPLE

$$A = \{1/a_1,\ 0.6/a_2, 0.3/a_3\},$$
$$B = \{0.6/b_1,\ 0.9/b_2, 0.1/b_3\}.$$

$$A \times B = \{\ 0.6/(a_1,b_1),\ 0.9/(a_1,b_2),\ 0.1/(a_1,b_3),\ 0.6/(a_2,b_1),\ 0.6/(a_2,b_2),\ 0.1/(a_2,b_3),\ 0.3/(a_3,b_1),\ 0.3/(a_3,b_2),\ 0.1/(a_3,b_3)\}$$

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

More generally, if $A_1, A_2, \ldots\ldots A_n$, *are* fuzzy subsets of $X_1, X_2, \ldots\ldots X_n$, then their cross product

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n,$$

is a fuzzy subset of

$$X_1 \times X_2 \times X_3 \times \ldots \times X_n, \text{ and}$$

$$\mu_T(x_1, x_2, x_3, \ldots.. x_n) = \underset{i}{MIN}\,[\mu_{A_i}(x_i)]$$

'Cross products' facilitate the mapping of fuzzy subsets that belong to disparate quantities or observations. This mapping is crucial for fuzzy rule based systems in general and fuzzy control systems in particular.

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

•Electric motors are used in a number of devices; indeed, it is impossible to think of a device in everyday use that does not convert electrical energy into mechanical energy – *air conditioners, elevators or lifts, central heating systems, …..*

•Electric motors are also examples of good control systems that run on simple heuristics relating to the speed of the (inside) rotor in the motor: change the strength of the magnetic field to adjust the speed at which the rotor is moving.

Electric motors can be electromagnetic and electrostatic; most electric motors are rotary but there are linear motors as well.

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

•Electric motors are also examples of good control systems that run on simple heuristics relating to the speed of the (inside) rotor in the motor:

If the motor is running too slow, then speed it up.
If motor speed is about right, then not much change is needed.
If motor speed is too fast, then slow it down.

INPUT: Note the use of reference fuzzy sets representing linguistic values *TOO SLOW, ABOUT RIGHT*, and, *TOO FAST*. The three linguistic values form the term set *SPEED*.

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

If the motor is running too slow, then speed it up.
If motor speed is about right, then not much change is needed.
If motor speed is too fast, then slow it down.

OUTPUT: In order to change speed, an operator of a control plant will have to apply more or less voltage: there are three reference fuzzy sets representing the linguistic values:

increase voltage (speed up);
no change (do nothing); and,
decrease voltage (slow down).

The three linguistic values for the term set *VOLTAGE.*

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

A fuzzy patch between the terms SPEED & VOLTAGE.

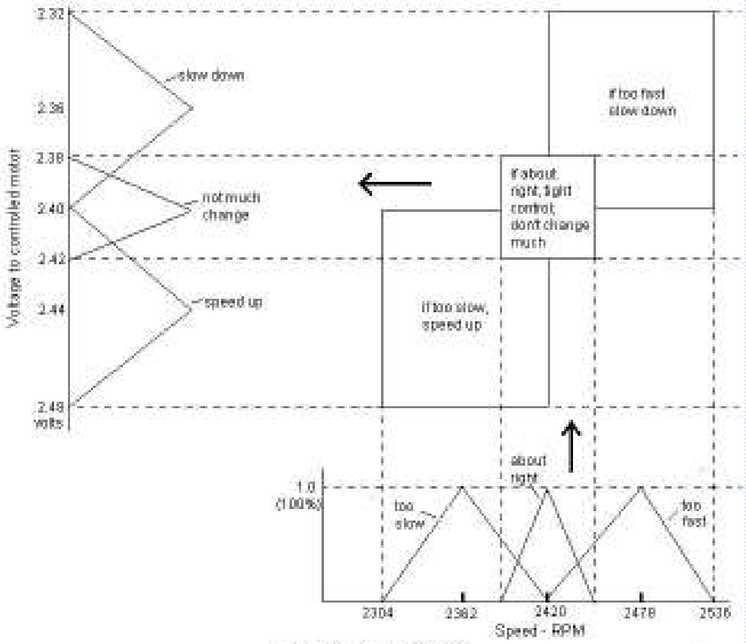
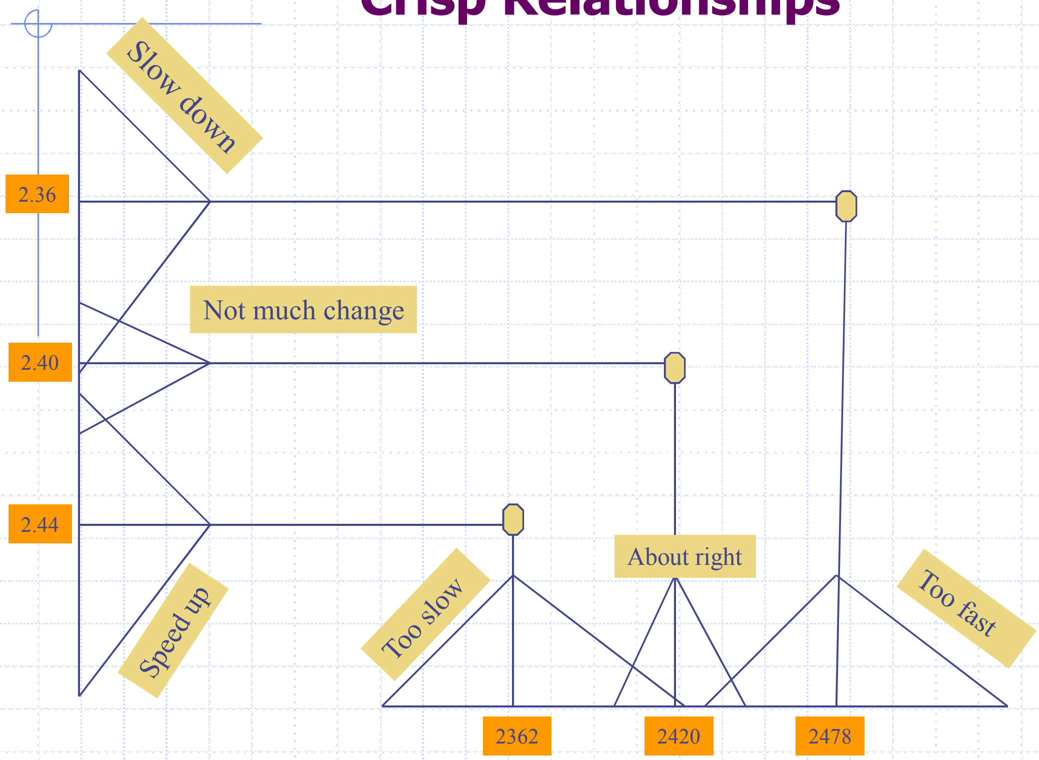


Figure 4 Cause-Effect

http://www.fuzzy-logic.com/ch3.htm

FUZZY LOGIC & FUZZY SYSTEMS

Crisp Relationships



FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE:
In order to understand how two fuzzy subsets are mapped onto each other to obtain a cross product, consider the example of an air-conditioning system. Air-conditioning involves the delivery of air which can be warmed or cooled and have its humidity raised or lowered.

An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has cooler and the cooler is under thermostatic control. Generally, the amount of air being compressed is proportional to the ambient temperature.

Consider Johnny's air-conditioner which has five control switches: ***COLD, COOL, PLEASANT, WARM*** **and** ***HOT.*** The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: ***MINIMAL, SLOW, MEDIUM, FAST*** **and** ***BLAST*****.**

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE:

The rules governing the air-conditioner are as follows:

RULE#1:	IF	TEMP *is* COLD	THEN	SPEED *is* MINIMAL
RULE#2:	IF	TEMP *is* COOL	THEN	SPEED *is* SLOW
RULE#3:	IF	TEMP *is* PLEASENT	THEN	SPEED *is* MEDIUM
RULE#4:	IF	TEMP *is* WARM	THEN	SPEED *is* FAST
RULE#5:	IF	TEMP *is* HOT	THEN	SPEED *is* BLAST

The rules can be expressed as a cross product:

$$\underline{CONTROL} = \underline{TEMP} \times \underline{SPEED}$$

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE:

The rules can be expressed as a cross product:

$$\underline{CONTROL} = \underline{TEMP} \times \underline{SPEED}$$

WHERE:

TEMP = {$\underline{COLD}$, $\underline{COOL}$, $\underline{PLEASANT}$, $\underline{WARM}$, $\underline{HOT}$}

SPEED = {$\underline{MINIMAL}$, $\underline{SLOW}$, $\underline{MEDIUM}$, $\underline{FAST}$, $\underline{BLAST}$}

$$\mu_{CONTROL}(T,V) = MIN[(\mu_{TEMP}(T), \mu_{SPEED}(V)]$$

$$RULE\#1: \quad IF \; 0 \leq T \leq 10^{\circ}C \quad \& \quad 0 \leq V \leq 30RPM$$

$$\mu_{CONTROL}(T,V) = MIN[(\mu_{TEMP}(T), \mu_{SPEED}(V)]$$

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE (CONTD.): **The temperature graduations are related to Johnny's perception of ambient temperatures:**

Temp (^{0}C).	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
17.5	N	Y	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE (CONTD.): **Johnny's perception of the speed of the motor is as follows:**

Rev/second (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	Y	N	N	N
20	Y	Y	N	N	N
30	Y	Y*	N	N	N
40	N	Y	Y	N	N
50	N	Y	Y*	N	N
60	N	N	Y	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE (CONTD.): The analytically expressed membership for the reference fuzzy subsets for the temperature are:

'COLD' $\mu_{COLD}(T) = \frac{-T}{10} + 1 \qquad 0 \le T \le 10;$

'COOL' $\mu^{(1)}_{COOL}(T) = \frac{T}{12.5} \qquad 0 \le T \le 12.5$

$\mu^{(2)}_{COOL}(T) = \frac{-T}{5} + 3.5 \qquad 12.5 \le T \le 17.5;$

'PLEASENT' $\mu^{(1)}_{PLEA}(T) = \frac{T}{2.5} - 6 \qquad 15 \le T \le 17.5$

$\mu^{(2)}_{PLEA}(T) = \frac{-T}{2.5} + 8 \qquad 17.5 \le T \le 20;$

'WARM'' $\mu^{(1)}_{WARM}(T) = \frac{T}{5} - 3.5 \qquad 17.5 \le T \le 22.5$

$\mu^{(2)}_{WARM}(T) = \frac{-T}{5} - 5.5 \qquad 22.5 \le T \le 27.5$

'HOT' $\mu^{(1)}_{HOT}(T) = \frac{T}{2.5} - 11 \qquad 25 \le T \le 30$

$\mu^{(2)}_{HOT}(T) = 1 \qquad T \ge 30$

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

Triangular membership functions can be described through the equations:

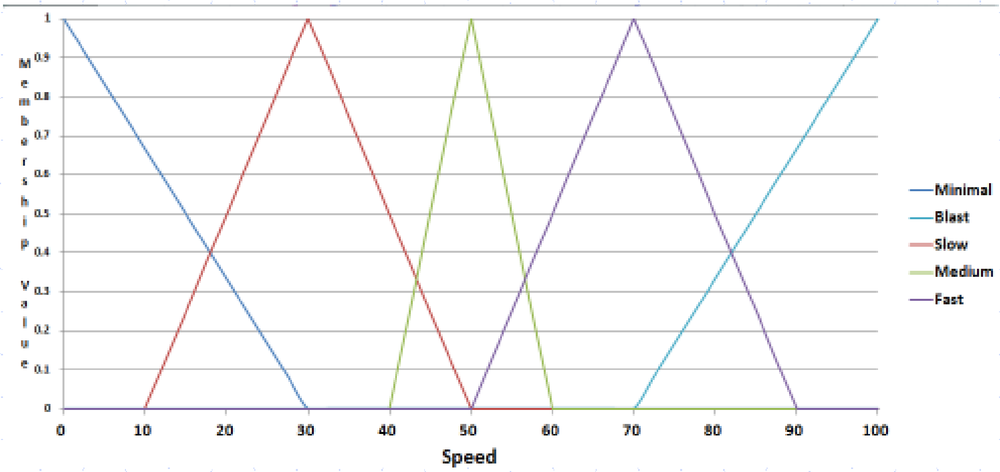
$$f(x;a,b,c) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & x \ge c \end{cases}$$

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

Triangular membership functions can be more elegantly and compactly expressed as

$$f(x;a,b,c) = \max(\min(\frac{x-a}{b-a},\frac{c-x}{c-b}),0)$$



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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

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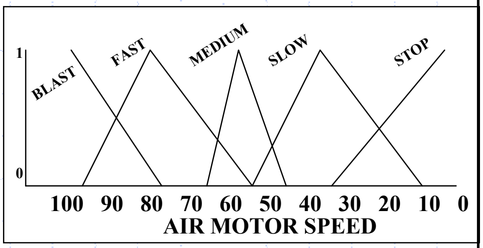

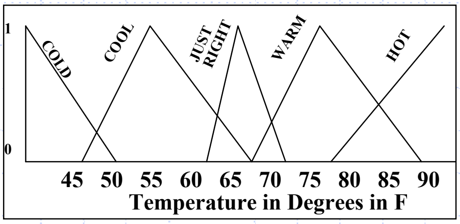

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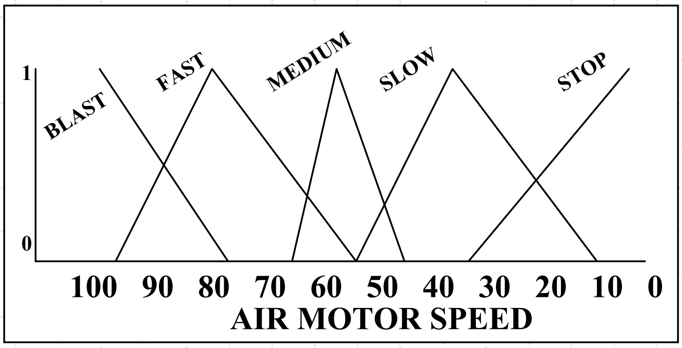
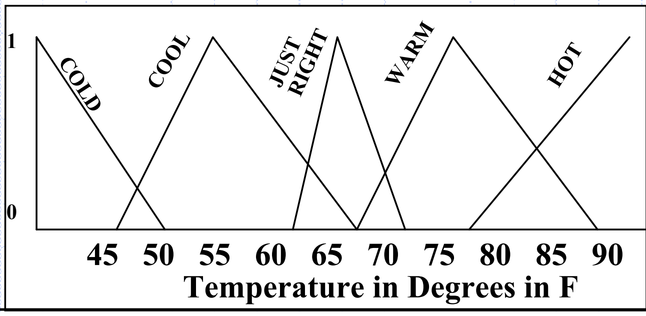
FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

A graphical representation of the two linguistic variables *Speed* and *Temperature*.





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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

EXAMPLE (CONTD.): The analytically expressed membership for the reference fuzzy subsets for speed are:

Term	Membership function	a	b	c
MINIMAL	$\mu_{MINIMAL}(V) = -\frac{V}{a} + c$	30		1
SLOW	$\mu_{SLOW}(V) = \max\left(\min\left(\frac{V-a}{b-a}, \frac{c-V}{c-b}\right), 0\right)$	10	30	50
MEDIUM	$\mu_{MEDIUM}(V) = \max\left(\min\left(\frac{V-a}{b-a}, \frac{c-V}{c-b}\right), 0\right)$	40	50	60
FAST	$\mu_{FAST}(V) = \max\left(\min\left(\frac{V-a}{b-a}, \frac{c-V}{c-b}\right), 0\right)$	50	70	90
BLAST	$\mu_{BLAST}(V) = \min\left(\frac{V-c}{a}, 1\right)$	30		70

Fuzzy Relationships

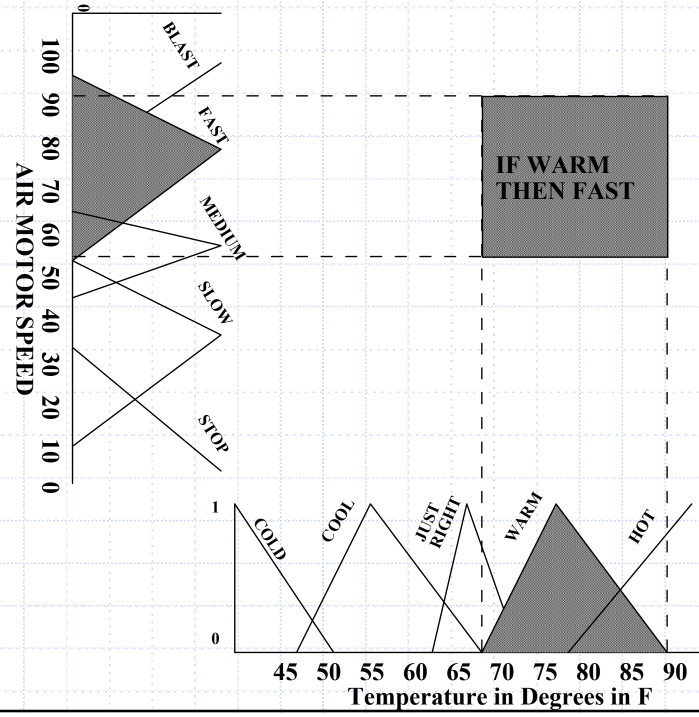
EXAMPLE (CONTD.): A sample computation of the SLOW membership function as a triangular membership function:

Speed (V)	$\left(\dfrac{V-a}{b-a}\right)$	$\left(\dfrac{c-V}{c-b}\right)$	$\mu_{SLOW}(V) = \max\left(\min\left(\dfrac{V-a}{b-a}, \dfrac{c-V}{c-b}\right), 0\right)$
10	0	2	0
15	0.25	1.75	0.25
20	0.5	1.5	0.5
25	0.75	1.25	0.75
30	1	1	1
35	1.25	0.75	0.75
40	1.5	0.5	0.5
45	1.75	0.25	0.25
50	2	0	0
55	2.25	-0.25	0

FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

A fuzzy patch is defined by a fuzzy rule: a patch is a mapping of two membership functions, it is a product of two geometrical objects, line segments, triangles, squares etc.



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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

In a fuzzy controller, a rule in the rule set of the controller can be visualized as a ‘device’ for generating the product of the input/output fuzzy sets.

Geometrically a patch is an area that represents the causal association between the cause (the inputs) and the effect (the outputs).

The size of the patch indicates the vagueness implicit in the rule as expressed through the membership functions of the inputs and outputs.

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FUZZY LOGIC & FUZZY SYSTEMS

Fuzzy Relationships

The total area occupied by a patch is an indication of the vagueness of a given rule that can be used to generate the patch.

Consider a one-input-one output rule: if the input is crisp and the output is fuzzy then the patch becomes a line. And, if both are crisp sets then the patch is vanishingly small – a point.

FUZZY LOGIC & FUZZY SYSTEMS

Recap → Fuzzy Sets

- A fuzzy set is an extension of the concept of a classical set whereby objects can be assigned partial membership of a fuzzy set; partial membership is not allowed in classical set theory.

- The degree an object belongs to a fuzzy set, which is a real number between 0 and 1, is called the membership value in the set.

- The meaning of a fuzzy set, is thus characterized by a *membership function* that maps elements of a universe of discourse to their corresponding membership values. The membership function of a fuzzy set *A* is denoted as μ .

Yen, John. (1998). Fuzzy Logic - A Modern Perspective (http://citeseer.ist.psu.edu/754863.html, site visited 16 October 2006)

FUZZY LOGIC & FUZZY SYSTEMS

Linguistic Terms and Variables

Zadeh has described the association between a fuzzy set and linguistic terms and linguistic variable.

Informally, a linguistic variable is a variable whose values are words or sentences in a natural or artificial language.

For example, if age is interpreted as a linguistic variable, then its term-set, T( ), that is, the set of its linguistic values, might be

T(age) = young + old + very young + not young + very old + very very young +rather young + more or less young + ...

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FUZZY LOGIC & FUZZY SYSTEMS

Linguistic Terms and Variables

Zadeh has described the association between a fuzzy set and linguistic terms and linguistic variable.

A primary fuzzy set, that is, a term whose meaning must be defined a priori, and which serves as a basis for the computation of the meaning of the non-primary terms in T( ). For example, the primary terms in

T(age) = young + old + very young + not young + very old + very very young +rather young + more or less young + ...

are young and old, whose meaning might be defined by their respective membership functions

$$\mu_{young} \quad and \quad \mu_{old}$$

Fuzzy Logic & Fuzzy Systems

Linguistic Terms and Variables

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are young and old, whose meaning might be defined by their respective membership functions

$$\mu_{young} \quad and \quad \mu_{old}$$

Non-primary membership functions

$\mu_{\text{very young}}$	$(\mu_{\text{young}})^2$
$\mu_{\text{more or less old}}$	$(\mu_{\text{old}})^{1/2}$
$\mu_{\text{not young}}$	$1-\mu_{\text{young}}$

FUZZY LOGIC & FUZZY SYSTEMS

Linguistic Terms and Variables

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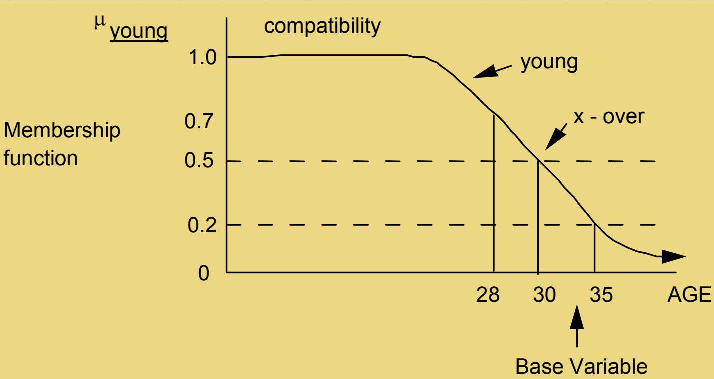
T(age) = young + old + very young + not young + very old + very very young +rather young + more or less young + ...

Primary fuzzy set –*young*- together with its cross-over point and linguistic or base variable



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FUZZY LOGIC & FUZZY SYSTEMS

Linguistic Terms and Variables

The association of a fuzzy set to a linguistic term offers the principal advantage in that human experts usually articulate their knowledge through the use of linguistic terms (*age, cold, warm…*). This articulation is typically comprehensible.

The followers of Zadeh have argued that advantage is reflected 'in significant savings in the cost of designing, modifying and maintaining a fuzzy logic system.' (Yen 1998:5)

Yen, John. (1998). Fuzzy Logic - A Modern Perspective (http://citeseer.ist.psu.edu/754863.html, site visited 16 October 2006)

FUZZY LOGIC & FUZZY SYSTEMS

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FUZZY LOGIC & FUZZY SYSTEMS

Computing with words?

Zadeh's notion of 'computing with words' (CWW) has been elaborated by Jerry Mendel:

CWW will provide 'a natural framework for humans to interact with computers using words, and that the computer would provide words back to the humans' (Mendel 2007:988).

CWW is effected by a computer program which allows inputs as words 'be transformed within the computer to "output" words, that are provided back to that human. CWW may take the form of IF–THEN rules, a fuzzy weighted average, a fuzzy Choquet integral, etc., for which the established mathematics of fuzzy sets provides the transformation from the input words to the output words.

Mendel, J. M. (2007) "Computing With Words and Its Relationships With Fuzzistics," *Information Sciences*, vol. 177, pp. 988-1006, 2007

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FUZZY LOGIC & FUZZY SYSTEMS

Union and intersection of fuzzy sets

The union preserves the commonalities as well as the differences across person FSs, whereas the intersection preserves only the commonalities.

Mendel, J. M. (2007) "Computing With Words and Its Relationships With Fuzzistics," *Information Sciences*, vol. 177, pp. 988-1006, 2007

Linguistic Terms and Variables

Two contrasting points about a *linguistic variable* are that it is a variable whose value can be interpreted quantitatively using a corresponding membership function, and **qualitatively** using an expression involving linguistic terms and

The notion of *linguistic variables* has led to a uniform framework where both qualitative and quantitative variables are used: some attribute the creation and refinement of this framework to be the reason that fuzzy logic is so popular as it is.

Yen, John. (1998). Fuzzy Logic - A Modern Perspective (http://citeseer.ist.psu.edu/754863.html, site visited 16 October 2006)