Characteristics of t-norms and fuzzy control

Andrea Mesiarová-Zemánková *

Khurshid Ahmad

Department of Computer Science

Trinity College

Dublin, IRELAND

mesiar@mat.savba.sk

Khurshid.Ahmad@cs.tcd.ie

Abstract

The application of conjunctive aggregation operators in fuzzy control systems with 2 inputs is discussed and the effect of the choice of a continuous t-norm operator in inference phase for TSK systems is computed. It is shown that the choice of a continuous t-norm modelling AND connective in antecedent part of fuzzy rules can be reduced just to strict or nilpotent t-norm. Moreover, it is shown that the isomorphism of strict (nilpotent) t-norms enable simpler fitting of TSK fuzzy systems parameters and reduces the computational complexity. Similar principle can be used also in the case of some non-commutative conjunctive aggregation operators modelling AND connective. The effect of the choice of a continuous t-norm is then evaluated on well known case studies in fuzzy control, the Sinc function and the urban traffic noise control system.

Keywords: t-norm, fuzzy system, fuzzy inference, isomorphism, t-conorm

*Permanent address: Mathematical Institute, Slovak Academy of Sciences, Bratislava, SLOVAKIA
1 Introduction

1.1 Background

There is an ever increasing number of applications of fuzzy control systems (FCS) [6, 7, 17, 23, 27, 28, 29, 36] ranging from the control of jet engines to theory of e-commerce systems and the specification and design of such systems pose interesting theoretical problems [9, 35]. The FCS are equally important for the development of fuzzy logic [2] and fuzzy control theory [7, 25, 31].

Typically, FCS are divided into model-free and model-based controllers, including Mamdani controllers [34], fuzzy slide mode controllers [5, 10], adaptive fuzzy controllers [16] and model-free controllers. TSK systems are exemplars of model-based controllers which are Lyapunov-stable [30, 32]. The neural fuzzy systems are in a class of their own due to the emergent properties associated with neural networks underpinning such system [24].

In the literature cited above, the choice of t-norms (and t-conorms) was typically that of a Product and Sum; and it is clear that fuzzy automata [2] may have different sets of properties if different t-norms (and t-conorms) were to be used; and the suitability of control structures of FCS can, in part, be determined by different choices of t-norms.

Advanced textbooks and monographs on neuro-fuzzy control systems [21], systems that are designed to learn the parameters of fuzzy membership functions and indeed fuzzy rules, suggest that 'instead of min another t-norm may be chosen, of course' [21, p:207]: However, this choice is seldom exercised although there are some exceptions here [3, 22]. Ciaramella et al. have dealt specifically with neuro-fuzzy system being based on a 'fuzzy relational ”IF-THEN” reasoning scheme' [3, p:146] (or model), and the authors have defined the structure of the model using 'different t-norms and t-conorms'. Ciaramella et al’s fuzzy-relational neural network model was implemented and was trained and tested to perform classification and approximation tasks – in both these tasks Ciaramella et al’s system usually outperforms the widely used multi-layer perceptrons or neural networks based on the radial-basis function.
architectures. The authors cite t-norms (and t-conorms) from Weber family together with the
Yager’s t-norms: however, apart from a note (and a graph) showing ‘classification [through] $B^1$ and $B^2$ fuzzy set estimation (sic)’ for the well-known IRIS database, the authors say little
about the different triangular norms they have used, they simply conclude by suggesting that
‘we obtain a 96% of correct classification on the training set.’ in ‘all [different norms] the
cases’ [3, p:156].

The aim of this paper is to explore whether the choice of a (continuous) t-norm has a
major effect on the output of a FCS. Note that this topic was so far investigated only in [4]
for so called crisp-type fuzzy logic controllers (which are nevertheless, up to some constraints,
similar to TSK systems), where the four basic t-norms were employed.

1.2 A note on triangular norms

A FCS system comprises linguistic rules and the interpretation of these rules in the compo-
sition and inference phases of the operation of the system involves the use of t-norms and
t-conorms (or, more generally, aggregation operators) for modelling intersection and union of
fuzzy sets [14]. In this paper we will focus only to TSK fuzzy systems, where the composition
is carried out by the weighted mean in which inputs are consequent constants and weights
are corresponding rule’s firing degrees. We will assume that the antecedent part of the rules
is connected by AND connectives only (note that OR connective can be always modelled
by an operator dual to the AND operator). Therefore in this paper we will focus just to
conjunctive operators, especially continuous t-norms.

A triangular norm (introduced in [18, 26]) is a binary operation $T : [0, 1]^2 \rightarrow [0, 1]$ which
is commutative, associative, non-decreasing in both variables and 1 is its neutral element.
Thus a t-norm is a special aggregation operator [1].

The literature on triangular norms suggests that there are families of (continuous) t-
norms, and their dual t-conorms – ranging from the familiar Gödelian minimum t-norm
(and the associated maximum t-conorm) to the more general Archimedean t-norms. A well-
grounded logical system should be based on a left-continuous t-norm. Many discontinuous t-norms can be approximated by continuous t-norms and therefore we focus only to continuous ones. To ensure some additional properties of a fuzzy system special classes of t-norms can be required (for example t-norms that satisfy some kind of Lipschitz property [19]).

Recall four widely cited basic t-norms [14]: Gödelian minimum $T_M : T_M(a, b) = \min(a, b)$, Product t-norm $T_P : T_P(a, b) = a \cdot b$, Lukasiewicz t-norm $T_L : T_L(a, b) = \max(0, a + b - 1)$ and Drastic product t-norm $T_D : T_D(a, b) = 0$ if $\max(x, y) < 1$, otherwise $T_D(a, b) = \min(a, b)$. Let us recall that $T_M$, $T_P$ and $T_L$ are 1-Lipschitz t-norms. Among most often used families of t-norms let us mention the Sugeno-Weber family of t-norms ([15]) given for $\lambda \in [-1, \infty]$ by $T^{SW}_\lambda(a, b) = \max\left(\frac{a+b-1+\lambda ab}{1+\lambda}, 0\right)$ with $T^{SW}_\infty = T_P$, $T^{SW}_{-1} = T_D$ and $T^{SW}_0 = T_L$. The Yager family of t-norms ([14]) is given for $\lambda \in [0, \infty]$ by $T^Y_\lambda(a, b) = \max(0, 1 - ((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}})$ with $T^Y_0 = T_D$, $T^Y_1 = T_L$ and $T^Y_\infty = T_M$. The Schweizer-Sklar family of t-norms ([14]) is given for $\lambda \in [-\infty, \infty]$ by $T^{SS}_\lambda(a, b) = \max(0, x^\lambda + y^\lambda - 1)^{\frac{1}{\lambda}}$. Here $T^{SS}_{-\infty} = T_M$, $T^{SS}_0 = T_P$, $T^{SS}_1 = T_L$ and $T^{SS}_{\infty} = T_D$. Note that all continuous t-norms are ordinal sums of continuous Archimedean t-norms and continuous Archimedean t-norms are either strict, i.e., isomorphic to the product t-norm $T_P$ or nilpotent, i.e., isomorphic to the Lukasiewicz t-norm $T_L$, (for more details see [14]).

In the following section we study the properties of a zero-order (first-order) TSK system where the basic t-norms are employed. We show that the choice of an aggregation operator modelling AND connective can be reduced to the class of strict (nilpotent) t-norms (Section 3). The boundaries for output of zero-order TSK system are computed (Section 4) and the effect of the choice made regarding t-norm operators on the output of TSK controller is shown on Case Studies (Section 5).
2 T-norm operators and zero-order Takagi-Sugeno-Kang system

In this section we will focus to zero-order TSK systems where the AND connective is modelled by a t-norm. We will analytically compute the output of the system and then show the effect of choosing one basic t-norm over another. Note that since for the drastic product t-norm for majority of inputs no rule fires we assume instead of the drastic product the nilpotent minimum t-norm $T_M^n$ given by

$$T_M^n(x, y) = \begin{cases} 
0 & \text{if } x + y \leq 1, \\
\min(x, y) & \text{else.}
\end{cases}$$

Consider that two-input-one-output zero-order TSK system is investigated. Let $x \in X$ and $y \in Y$ denote the input variables and $z \in Z$ the output variable with term sets $\{A_1, A_2\}$ for $x$ and $\{B_1, B_2\}$ for $y$ (which are possibly the subsets of larger term sets). The four possible rules are in the table below.

<table>
<thead>
<tr>
<th>$y$ \ $x$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>$k_3$</td>
<td>$k_3$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$k_2$</td>
<td>$k_4$</td>
</tr>
</tbody>
</table>

Table 1: Four rules for zero-order TSK system

We will consider first the cases where at any given time only two rules in a rule-base are invoked. Up to the symmetry there are two possible rule-bases with two rules. Note that in the following $T$ will be a t-norm and $a_1, a_2$ ($b_1, b_2$) will be the membership degrees of $x$ ($y$) in $A_1, A_2$ ($B_1, B_2$), respectively.

The first is the rule base (bold in Table 1)
$TSII(1):$ If $x$ is $A_1$ AND $y$ is $B_1$ then $z = k_1$
If $x$ is $A_2$ AND $y$ is $B_2$ then $z = k_4$

The output of this system, if at least one rule fires, will be

$$\frac{T(a_1, b_1)k_1 + T(a_2, b_2)k_4}{T(a_1, b_1) + T(a_2, b_2)}.$$ 

Thus when $T(a_1, b_1) = T(a_2, b_2)$ then the output will be $\frac{k_1+k_4}{2}$.

Second is the rule base (italic in Table 1)

$TSII(2):$ If $x$ is $A_1$ AND $y$ is $B_1$ then $z = k_1$
If $x$ is $A_1$ AND $y$ is $B_2$ then $z = k_2$

The output of this system, if at least one rule fires, will be

$$\frac{T(a_1, b_1)k_1 + T(a_1, b_2)k_2}{T(a_1, b_1) + T(a_1, b_2)}.$$ 

Thus when $T(a_1, b_1) = T(a_1, b_2)$ then the output will be $\frac{k_1+k_2}{2}$.

Finally, for whole rule-base TSIV the output of the system is given by

$$\frac{T(a_1, b_1)k_1 + T(a_1, b_2)k_2 + T(a_2, b_1)k_3 + T(a_2, b_2)k_4}{T(a_1, b_1) + T(a_1, b_2) + T(a_2, b_1) + T(a_2, b_2)}.$$ 

If we assume that fuzzy sets $A_1$ and $A_2$ ($B_1$ and $B_2$) define a Ruspini partition, i.e. $a_1 + a_2 = 1$ ($b_1 + b_2 = 1$) this assumption will simplify the analytical description of the output (difference in output) for each of the four t-norms for $TSII(1)$ ($TSII(2)$, $TSIV$). These descriptions can be found in Tables 2 and 3.
### Rule base TSII(1)

<table>
<thead>
<tr>
<th>T-norm</th>
<th>$a + b &lt; 1$</th>
<th>$a + b = 1$</th>
<th>$a + b &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>$a \leq b$</td>
<td>$a &lt; b$</td>
<td>$a &gt; b$</td>
</tr>
<tr>
<td>$T_L$</td>
<td>$k_4$</td>
<td>$\frac{k_1+k_4}{2}$</td>
<td>$k_1$</td>
</tr>
<tr>
<td>$T_{M^*}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$T_P$</td>
<td>$\frac{(k_1-k_4)(a-b)}{2ab+1-a-b}$</td>
<td>$0$</td>
<td>$\frac{(k_1-k_4)(1-a)(1-b)}{2ab+1-a-b}$</td>
</tr>
<tr>
<td>$T_M$</td>
<td>$\frac{k_1-k_4}{2}$</td>
<td>$\frac{k_1-k_4}{2}$</td>
<td>$\frac{k_1-k_4}{2}$</td>
</tr>
</tbody>
</table>

### Rule base TSII(2)

<table>
<thead>
<tr>
<th>T-norm</th>
<th>$a + b &lt; 1$</th>
<th>$a + b = 1$</th>
<th>$a + b &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>$\frac{k_1+k_2}{2}$</td>
<td>$k_2$</td>
<td>$k_2$</td>
</tr>
<tr>
<td>$T_L$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$T_{M^*}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$T_P$</td>
<td>$\frac{(k_1-k_2)(2b-1)}{2}$</td>
<td>$(k_1-k_2)b$</td>
<td>$(k_1-k_2)(1-b)$</td>
</tr>
<tr>
<td>$T_M$</td>
<td>$0$</td>
<td>$\frac{(k_1-k_2)b}{a+b}$</td>
<td>$\frac{(k_1-k_2)b}{a+b}$</td>
</tr>
</tbody>
</table>

Table 2: Analytical description of differences of outputs for rule bases TSII(1), TSII(2) from a common baseline for t-norms $T_L$, $T_{M^*}$, $T_P$ and $T_M$.

Point to note here: Use of different t-norms as AND operators yield distinct characteristics of the resulting output function. For example in the case of two-rules rule bases nilpotent t-norms always yield a function which is piecewise linear(constant). This information can, depending on the type of input membership function, help to identify a t-norm from real data. As it can be seen from Tables 2 and 3 one disadvantage of nilpotent t-norms, or more precisely, of the t-norms with zero divisors is that when we model AND connective by such a t-norm then for several inputs no rule is fired. This cause the non-continuity of the output with respect to input variables in the area around the input in which no rule is fired. Therefore, when modelling a control problem with nilpotent t-norms, in several areas our model can have some kind of "switching mode" behavior.

A nilpotent minimum is a non-continuous t-norm, however, it is a very important "prototypical" t-norm with both nilpotent and idempotent elements and thus, although we want to focus mainly to continuous t-norms, we introduce results also for the nilpotent minimum t-norm. Note that the question whether nilpotent minimum can be approximated by continuous t-norms is still an open problem.
### Rule base TSIV

| Lukasiewicz $T_L$ | $\frac{k_4 - a(k_1 + k_4) + b(k_3 - k_4)}{k_4 + a(k_2 - k_1 + b(k_2 + k_1))}$ | $a + b < 1, a < b$
|-------------------|---------------------------------|-----------------
|                   | $\frac{1 - 2b}{1 - 2a}$         | $a + b < 1, a \geq b$
|                   | $-k_1 + a(k_1 - k_3) + b(k_1 + k_3)$ | $a + b < 1, a < b$
|                   | $\frac{k_1 + a(k_1 + k_2) + b(k_1 - k_2)}{2b - 1}$ | $a + b \geq 1, a < b$
|                   | $-k_1 + a(k_1 + k_2) + b(k_1 - k_2)$ | $a + b \geq 1, a < b$
|                   | $\frac{2a - 1}{2a - 1}$         | $a + b > 1, a = b$
|                   | $\emptyset$                      | $a + b = 1, a = b$

| Nilpotent minimum $T_M^a$ | $k_3 b + k_4 (1 - b)$ | $a + b < 1, a < b$
|---------------------------|----------------------|-----------------
|                            | $k_2 a + k_4 (1 - a)$ | $a + b < 1, a > b$
|                            | $k_1 a + k_3 (1 - a)$ | $a + b > 1, a < b$
|                            | $k_1 b + k_2 (1 - b)$ | $a + b > 1, a > b$
|                            | $k_4$                 | $a + b > 1, a = b$
|                            | $k_3$                 | $a + b = 1, a < b$
|                            | $\emptyset$           | $a + b = 1, a = b$
|                            | $k_2$                 | $a + b = 1, a > b$
|                            | $k_1$                 | $a + b > 1, a = b$

| Product $T_P$ | $a b (k_1 - k_2 - k_3 + k_4) + a (k_2 - k_4) + b (k_3 - k_4) + k_1$ | $a + b < 1, a < b$

| Minimum $T_M$ | $\frac{k_4 + a(k_1 + k_2) + b(k_3 - k_4)}{2a + b}$ | $a + b < 1, a < b$
|---------------|---------------------------------|-----------------
|               | $\frac{2a + 1}{k_1 + a(k_2 - k_1) + b(k_1 + k_3)}$ | $a + b < 1, a \geq b$
|               | $\frac{2b + 1}{k_2 + k_3 + a(k_1 - k_3) - b(k_2 + k_4)}$ | $a + b \geq 1, a < b$
|               | $\frac{3 - 2b}{3 - 2a}$ | $a + b \geq 1, a \geq b$

|                   | $\emptyset$ | $a + b = 1, a = b$

Table 3: Analytical description of output of rule base TSIV (where all four rules from Table 1 are assumed) for t-norms $T_L$, $T_M^a$, $T_P$ and $T_M$.

### 2.1 A note on two-rules rule bases

In both rule-bases TSII(1) and TSII(2) we see that for any input the output is a weighted mean of the constants $k_1$ and $k_2$ ($k_4$), where the weights depend on the input $(x, y)$ (i.e., weights are functions of variables $x$ and $y$). Let us denote by $Out_{k_1,k_2}(x, y)$ the output of the system (for TSII(1) or TSII(2)) corresponding to some input $(x, y)$. Then $Out_{k_1,k_2}(x, y) = w_{x,y}^1 k_1 + w_{x,y}^2 k_2$. Assume $k_1 \leq k_2$. Since weighted means are shift invariant and homogenous we know that $Out_{k_1,k_2}(x, y) = k_1 + Out_{0,k_2-k_1}(x, y) = k_1 + \frac{k_2 - k_1}{2} Out_{0,2}(x, y) = \frac{k_1 + k_2}{2} + \frac{k_2 - k_1}{2} Out_{-1,1}(x, y)$. Thus this situation can be always transformed to the situation when $k_1 = -1$, $k_2 = 1$. For such consequent values we can compare how does the dependence of output on fuzzy membership degrees (in the case of Ruspini partition) differ for different
Figure 1: Output of a zero-order TSK system with rule base TSII(1), depending on $a = \mu_A(x)$ and $b = \mu_B(y)$ with $k_1 = 1$ and $k_4 = -1$; for the t-norm $T_L$ and $T_{M^*}$ (left), $T_P$ (right), $T_M$ (bottom).

t-norms simply by computing

$$\text{Dif}_{T_1, T_2} = \int_0^1 \int_0^1 |\text{Out}_{T_1}(a_1, b_1) - \text{Out}_{T_2}(a_1, b_1)| da_1 db_1.$$  

The results of this comparison are can be found in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>Rule base TIII(1)</th>
<th>Rule base TIII(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Dif}_{T_P, T_M}$</td>
<td>0.1137</td>
<td>0.1667</td>
</tr>
<tr>
<td>$\text{Dif}_{T_P, T_L}$</td>
<td>0.3863</td>
<td>0.2917</td>
</tr>
<tr>
<td>$\text{Dif}<em>{T_P, T</em>{M^*}}$</td>
<td>0.3863</td>
<td>0.25</td>
</tr>
<tr>
<td>$\text{Dif}_{T_L, T_M}$</td>
<td>0.5</td>
<td>0.2917</td>
</tr>
<tr>
<td>$\text{Dif}<em>{T_L, T</em>{M^*}}$</td>
<td>0</td>
<td>0.0417</td>
</tr>
<tr>
<td>$\text{Dif}<em>{T</em>{M^*}, T_M}$</td>
<td>0.5</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 4: Overall difference between outputs of zero-order TSK fuzzy system with $k_1 = 1$ and $k_2 = k_4 = -1$. We assume $\text{Out}_T(x, y) = 0$ – as a kind of neutral output – in the case when no rule fires.
Figure 2: Output of a zero-order TSK system with rule base TSII(2), depending on $a = \mu_A(x)$ and $b = \mu_B(y)$ with $k_1 = 1$ and $k_4 = -1$; for the t-norm $T_L$ (top left) and $T_M^n$ (top right), $T_P$ (bottom left), $T_M$ (bottom right).

2.2 Relationship between zero-order and first-order TSK system

Assume a two-input-one-output first-order TSK model with term sets $\{A_1, A_2\}$ for input variable $x$ and $\{B_1, B_2\}$ for input variable $y$. Let us focus on the area, were for every input $(x, y)$ the same two rules are fired. Since for all such rule-bases the situation is similar we will investigate just one on them, for example:

$$TSII: \begin{align*}
\text{If } x \text{ is } A_1 \text{ AND } y \text{ is } B_1 \text{ then } z &= k_1 x + q_1 y + r_1 \\
\text{If } x \text{ is } A_2 \text{ AND } y \text{ is } B_2 \text{ then } z &= k_2 x + q_2 y + r_2
\end{align*}$$

Let AND connective be modelled by a t-norm $T$. For input $(x, y)$ denote $a_1 = \mu_{A_1}(x)$, $a_2 = \mu_{A_2}(x)$, and $b_1 = \mu_{B_1}(y)$, $b_2 = \mu_{B_2}(y)$. Then

$$Out_{k,q,r}(x, y) = \frac{T(a_1, b_1)(k_1 x + q_1 y + r_1) + T(a_2, b_2)(k_2 x + q_2 y + r_2)}{T(a_1, b_1) + T(a_2, b_2)}.$$
Figure 3: Output of a zero-order TSK system with rule base TSIV, depending on $a = \mu_A(x)$ and $b = \mu_B(y)$ with $k_1 = -1$, $k_2 = -2$, $k_3 = 3$ and $k_4 = 1$; for the t-norm $T_L$ (top, left), $T_M^-$ (top, right), $T_P$ (bottom, left), $T_M^+$ (bottom, right).

For an input $(x, y)$ let us denote by $Out_r(x, y)$ the output of the zero-order TSK model with the same input partitions and the same rule antecedents as our first-order TSK model, where the consequents of the two rules are constants $r_1, r_2$, respectively. Then $Out_r(x, y) = T(a_1, b_1) r_1 + T(a_2, b_2) r_2$. Thus we get

$$Out_{k,q,r}(x, y) = Out_r(x, y) + Out_k(x, y) x + Out_q(x, y) y.$$ 

It is evident that we can obtain similar expression for any rule base (e.g. TSIV) which is common (up to consequents) to some first-order and zero-order TSK model. The same can be also done for TSK models of higher orders.

As we have mentioned above for all zero-order TSK models with two rules we have $Out_r(x, y) = \frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2} Out_{-1,1}(x, y)$, where for an input $(x, y)$, $Out_{-1,1}(x, y)$ is output of a zero-order TSK model with the same two rules and the same partitions of input space as our first-order TSK model where the consequence constants are $-1, 1$, respectively. Together
we get \( \text{Out}_{k,q,r}(x,y) = \frac{r_1 + r_2}{2} + \frac{k_1 + k_2}{2} x + \frac{q_1 + q_2}{2} y + \text{Out}_{-1,1}(x,y) \left( \frac{r_2 - r_1}{2} + \frac{k_2 - k_1}{2} x + \frac{q_2 - q_1}{2} y \right) \), i.e., we have \( \text{Out}_{-1,1}(x,y) = \frac{\text{Out}_{k,q,r}(x,y) - (r_1 + r_2 + k_1 + k_2 x + q_1 + q_2 y)}{r_2 - r_1 + \frac{k_2 - k_1}{2} x + \frac{q_2 - q_1}{2} y} \). This means that if consequence constants \( k_1, k_2, q_1, q_2, r_1, r_2 \) are fixed, then to tune our fuzzy sets and AND operator it is enough to fit the zero order TSK model with two rules and consequents constants \(-1, 1\) in order to model a function

\[
G(x, y) = \frac{F(x, y) - \left( \frac{r_1 + r_2}{2} + \frac{k_1 + k_2}{2} x + \frac{q_1 + q_2}{2} y \right)}{\frac{r_2 - r_1}{2} + \frac{k_2 - k_1}{2} x + \frac{q_2 - q_1}{2} y}.
\]

3 The AND operator in zero-order TSK fuzzy system

T-norms and t-conorms are extensively used in the fuzzy inference. Usually a t-norm (t-conorm) is used to model AND (OR) connective in the antecedent of a fuzzy rule. T-conorms are used as aggregation operators, which aggregate outputs of different fuzzy rules in order to get just one output.

Some authors generalize t-norms in order to model AND connective. For example uninorms which generalize t-norms and have compensatory behavior (which means that the small membership degree of one input can be compensated by higher membership degree of second input) are used by several authors [20, 33] in fuzzy logic. Note that uninorm is an associative, commutative, non-decreasing binary operation which acts on unit square and have a neutral element \( e \in [0, 1] \). Although uninorms generalize t-norms, these operations are again associative even though this property is not needed. That is why while modelling AND connective, one has to ask what are the properties that the operator has to satisfy?

In practice, the AND operator is usually required to be monotonic – if for an input its corresponding belongingness degrees are largest then its firing degree (FD) should be highest as compared to the firing degree associated other inputs that have lower degrees of belongingness.

It is also reasonable to expect, that 0,1 are idempotent elements of AND operator – if
for an input its corresponding belongingness degrees are 0 (1) then its firing degree should also be 0 (1). Thus the properties which should be satisfied by AND operator are those of aggregation operators.

An AND operator can be required to have some additional properties. For example:

Anihilator property – \( A(0, b) = A(a, 0) = 0 \), – if any part of the antecedent is completely unsatisfied than the firing degree should be zero.

Boundedness by minimum – \( A(a, b) \leq \min(a, b) \) – may be required in order to ensure that the firing degree will not exceed any of the membership values. However, in such a case AND operator is not compensatory.

It can be argued that the other properties of t-norms – associativity, commutativity and the existence of neutral element are not as essential for the AND operator. Indeed, in many practical cases, it is preferable to ignore the commutativity if one input is of greater importance than another. The associativity of AND operator ensures that if a new term is added to the antecedent of the rule then the new firing degree of the rule is easily computed from the old firing degree. However, the associativity of AND operator does not seem to be essential for the performance of the fuzzy system.

Before we will be able to discuss the more general AND operators, we have to examine the basic case of continuous t-norms in more detail. Several ideas from this investigation can be then extended to more general operators (see Subsection 3.3).

In the following subsection we will show how can be the selection of a continuous t-norm for modelling of AND connective reduced just to selection of a proper isomorphism. We will show that for t-norms which are very close also the output of zero-order TSK system will be close. Since strict (as well as nilpotent) t-norms are dense in the class of continuous t-norms it is enough to reduce the selection of a t-norm to one of these classes. The isomorphism in the class of strict (nilpotent) t-norms ensures further simplification.
3.1 Output of the TSK system for \( \varepsilon \)-close \( t \)-norms

It is known [14] that every continuous \( t \)-norm \( T \) can be approximated with arbitrary precision by a strict \( t \)-norm (as well as by a nilpotent \( t \)-norm) i.e., for all \( \varepsilon > 0 \) there exist a strict \( t \)-norm \( T_\varepsilon \) such that \( T \) and \( T_\varepsilon \) are \( \varepsilon \)-close, i.e., for all \( (a, b) \in [0, 1]^2 \) we have \( |T(a, b) - T_\varepsilon(a, b)| < \varepsilon \). If we assume the rule base TSII(1) and an input \((x, y)\) with \( \mu_{A_i}(x) = a \) and \( \mu_{B_i}(y) = b \) (with Ruspini partition) then \( |\text{Out}_T - \text{Out}_{T_\varepsilon}| < \frac{\varepsilon|k_1 - k_2|}{T(a, b) + T_\varepsilon(a, 1 - b)} \). Similarly for rule base TSII(2) we get \( |\text{Out}_T - \text{Out}_{T_\varepsilon}| < \frac{\varepsilon|k_1 - k_2|}{T(a, b) + T(a, 1 - b)} \). Further if we assume a rule base TSIV and an input \((x, y)\) with \( \mu_{A_i}(x) = a \) and \( \mu_{B_i}(y) = b \) then

\[
|\text{Out}_T - \text{Out}_{T_\varepsilon}| < \frac{3\varepsilon \max_{i,j} |k_i - k_j|}{T(a, b) + T(1 - a, b) + T(a, 1 - b) + T(1 - a, 1 - b)}
\]

as well as

\[
|\text{Out}_T - \text{Out}_{T_\varepsilon}| < \frac{3\varepsilon \max_{i,j} |k_i - k_j|}{T_\varepsilon(a, b) + T_\varepsilon(1 - a, b) + T_\varepsilon(a, 1 - b) + T_\varepsilon(1 - a, 1 - b)}
\]

which means that whenever \( T(a, b) + T(1 - a, b) + T(a, 1 - b) + T(1 - a, 1 - b) \geq r > 0 \) \( (T_\varepsilon(a, b) + T_\varepsilon(1 - a, b) + T_\varepsilon(a, 1 - b) + T_\varepsilon(1 - a, 1 - b) \geq r > 0) \) for all possible inputs then for arbitrary \( \delta > 0 \) we can find a \( t \)-norm \( T_\delta \) such that \( \text{Out}_T \) and \( \text{Out}_{T_\delta} \) are \( \delta \)-close (i.e., absolute value of difference between \( \text{Out}_T \) and \( \text{Out}_{T_\delta} \) is smaller then \( \delta \) for all considered inputs).

However, \( T(a, b) + T(1 - a, b) + T(a, 1 - b) + T(1 - a, 1 - b) \geq r > 0 \) is not satisfied for all \( (a, b) \in [0, 1]^2 \) by nilpotent \( t \)-norms. Thus the problem arises when two nilpotent \( t \)-norms are compared. However, the majority of real problems work on a discrete scale and thus this problem can be solved by proper choice of fuzzy sets. For example, \( t \)-norms \( T_P \) and \( T_L \) are \( \frac{1}{4} \)-close, which means that for any input in a rule base TSIV (with four rules) the difference between \( \text{Out}_{T_L}(x, y) \) and \( \text{Out}_P(x, y) \) is smaller than \( \frac{3}{4} \max_{i,j} |k_i - k_j| \). Note that number of non-continuous \( t \)-norms can be approximated by strict (nilpotent) \( t \)-norms as well.

The above result shows us that it is always enough to find the best strict (nilpotent)
t-norm to model AND connective. However, even in the class of strict (nilpotent) t-norms, the selection of the best AND operator is not easy. In general, it is impossible to say which class of t-norms (strict or nilpotent) should be used for the selection of AND operator. One way to decide could be to check the fit of the exemplar t-norms, i.e., the product $T_P$ and the Lukasiewicz t-norm $T_L$ first, and focus to strict (nilpotent) t-norms in the case that $T_P$ ($T_L$) gives the better fit.

### 3.2 Isomorphism of t-norms

In this subsection we will focus just to antecedent part and firing degrees of the rules. Consider a rule of zero-order TSK fuzzy system with two inputs, for example $R_1$:

$$R_1 : \text{If } x \text{ is } A \text{ AND } y \text{ is } B \text{ then } z = k.$$  

Here, $A$ and $B$ are modelled by membership functions $\mu_A(x)$ and $\mu_B(y)$, respectively. If we chose a strict (nilpotent) t-norm $T_1$ as AND operator, the firing degree of $R_1$, for input $(x, y)$, is:

$$\text{FD}(R_1) = T_1(\mu_A(x), \mu_B(y)). \quad (1)$$

As we have mentioned in Subsection 1.2, each strict t-norm is isomorphic to the product t-norm $T_P$ and each nilpotent t-norm is isomorphic to the Lukasiewicz t-norm $T_L$. Using this isomorphic relations, we can rewrite (1) as:

$$\text{FD}(R_1) = f^{-1}(f(\mu_A(x))f(\mu_B(x))) \quad (2)$$

in the case of strict t-norms and as:

$$\text{FD}(R_1) = f^{-1}(T_L(f(\mu_A(x)), f(\mu_B(x)))) \quad (3)$$
in the case of nilpotent t-norms, where \( f: [0, 1] \rightarrow [0, 1] \) is an isomorphism. Let us define fuzzy sets \( C \) and \( D \) by \( \mu_C(x) = f(\mu_A(x)) \), \( \mu_D(y) = f(\mu_B(y)) \) and recall that \( f^{-1} \) is also an isomorphism. The definition of an isomorphism \( g: [0, 1] \rightarrow [0, 1] \) by \( g = f^{-1} \) results in the final expression:

\[
FD(R_1) = g(\mu_C(x)\mu_D(x)) \tag{4}
\]

for strict and

\[
FD(R_1) = g(T_L(\mu_C(x), \mu_D(x))) \tag{5}
\]

for nilpotent t-norms. The equations (1), (2) and (4) ((3), (5)) show that the empirical approach of selecting fuzzy sets and a strict (nilpotent) t-norm is equivalent to the search for an isomorphism \( g \) and the fuzzy sets \( (C \) and \( D \) for example). Furthermore, while changing the isomorphism \( g \) to get a better fit the effect of this change is more readable than the effect of the change of a t-norm \( T_1 \).

To reduce the system complexity the isomorphism \( g \) is preferably defined by a small number of parameters. Power transformations \( g(x) = x^p \) for \( p \in [0, \infty[ \) play an important role between the isomorphisms defined by just one parameter. Another interesting class of isomorphisms are polynomial functions. Quadratic isomorphisms are defined by one parameter, as well. Here \( g(x) = px^2 + (1 - p)x \) with \( p \in [-1, 1] \). Among isomorphisms defined by two parameters let us recall piecewise linear functions

\[
g(x) = \begin{cases} 
\frac{q}{p}x & \text{if } x < p, \\
\frac{(1-q)x+q-p}{1-p} & \text{else.}
\end{cases}
\]

with \( p, q \in ]0, 1[ \). We will show the effect of such isomorphisms in the Case Study in Subsection 5.1.
3.3 Non-commutative rule antecedents

Assume an antecedent of a fuzzy rule:

\[
\text{If } x \text{ is } A \text{ AND } y \text{ is } B.
\]

In the case that the first part of this antecedent "\(x \text{ is } A\)" is more important than the second part "\(y \text{ is } B\)" for achieving a greater firing degree, the AND operator will no longer be commutative, i.e., it cannot be a t-norm. However, for \(a = \mu_A(x)\) and \(b = \mu_B(x)\), this situation can be modelled by the firing degree given by

\[
\text{FD}(a, b) = T(a^2, b).
\]

Thus AND operator is no longer commutative (nor associative and it has no neutral element). It is clear that the membership degree \(a\) is now more important than the membership degree \(b\). For example, assume \(a = 0.5\) and \(b = 1\), i.e., first part of the antecedent is half fulfilled and second part is totally fulfilled. Then the firing degree \(\text{FD}(0.5, 1) = T(0.25, 1) = 0.25\).

However, in the symmetric situation, when \(a = 1\) and \(b = 0.5\), i.e., when first part is totally satisfied and the second part is half satisfied, the firing degree is given by \(\text{FD}(1, 0.5) = T(1, 0.5) = 0.5\). Note that the same result can be achieved by the simple redefinition of the fuzzy set \(A\). Consider a fuzzy set \(A'\) given by \(\mu_{A'}(x) = (\mu_A(x))^2\). Then the rule with antecedent

\[
\text{If } x \text{ is } A' \text{ AND } y \text{ is } B
\]

gives the same firing degree as the original rule.

We will now show that a number of situations, where non-commutative operators are necessary, can be modelled by firing degree \(T(f(a), g(b))\), where \(T\) is a t-norm and \(f, g\) are increasing bijections on \([0, 1]\). As we have mentioned before, the firing degree of the rule should be modelled by an aggregation operator. Note that for every continuous generated
aggregation operator $A$ such that $A(a,0) = A(0,b) = 0$ and $A(a,b) \leq \min(a,b)$ there exist such strictly decreasing continuous functions $s, p, q$ that $A(a,b) = s(p(a) + q(b))$ (see [13]). Therefore

$$FD(a,b) = s(p(a) + q(b)).$$

Assume functions $h, f, g : [0,1] \rightarrow [0,1]$ given by $h(x) = e^{-s^{-1}(x)}, f(x) = h^{-1}(e^{-p(x)})$ and $g(x) = h^{-1}(e^{-q(x)})$. Then

$$s(p(a) + q(b)) = h^{-1}(e^{-(-\ln(h(f(a))) - \ln(h(g(b))))}) = h^{-1}(h(f(a))h(g(b))) = T(f(a), g(b))$$

for some strict t-norm $T$. Hence we see that the same firing degree can be achieved if we model AND connective by the t-norm $T$ and use the rule antecedent

$$\text{If } x \text{ is } A' \text{ AND } y \text{ is } B',$$

where $\mu_{A'}(x) = f(\mu_A(x))$ and $\mu_{B'}(y) = g(\mu_B(y))$.

## 4 Output boundaries for continuous t-norms in zero-order TSK systems

In this section we will investigate the boundaries of possible outputs of a zero-order TSK fuzzy system when different continuous t-norms are applied. We will assume the rule base TIV with all four rules from Section 2. Assume input $(x, y)$ and recall that $a_1 = \mu_{A_1}(x)$ ($a_2 = \mu_{A_2}(x)$) and $b_1 = \mu_{B_1}(x)$ ($b_2 = \mu_{B_2}(x)$). First let us note that without loose of generality we can assume $a_1 \leq a_2$ and $b_1 \leq b_2$. First we will introduce the following important proposition:

### Proposition 1

Let $0 \leq c_1, c_2, c_3, c_4 \leq 1$ with $c_1 + c_2 + c_3 + c_4 = 1$ and let $k_1, k_2, k_3, k_4 \in \mathbb{R}$. Denote $A_j = \{k_i \mid i \in \{1, 2, 3, 4\}, c_i \geq c_j\}$ for $j = 1, 2, 3, 4$. Further let $F_j = \sum_{x \in A_j} x / |A_j|$, where $|A_j|$ denotes
the number of elements in the set $A_j$. Then $c_1 k_1 + c_2 k_2 + c_3 k_3 + c_4 k_4 \in \left[ \min_{j=1,2,3,4} F_j, \max_{j=1,2,3,4} F_j \right]$.

The proof of this proposition can be found in Appendix. Note that

$$c(1) k(1) + c(2) k(2) + c(3) k(3) + c(4) k(4) =$$

$$\begin{cases}
  k(1) & \text{if } c(1) = 1, c(2) = 0, c(3) = 0, c(4) = 0, \\
  \frac{k(1)+k(2)}{2} & \text{if } c(1) = \frac{1}{3}, c(2) = \frac{1}{2}, c(3) = 0, c(4) = 0, \\
  \frac{k(1)+k(2)+k(3)}{3} & \text{if } c(1) = \frac{1}{3}, c(2) = \frac{1}{3}, c(3) = \frac{1}{3}, c(4) = 0, \\
  \frac{k(1)+k(2)+k(3)+k(4)}{4} & \text{if } c(1) = \frac{1}{4}, c(2) = \frac{1}{4}, c(3) = \frac{1}{4}, c(4) = \frac{1}{4}.
\end{cases}$$

Recall that we suppose $a_1 \leq a_2$ and $b_1 \leq b_2$ and therefore $T(a_1, b_1) \leq T(a_1, b_2) \leq T(a_2, b_2)$ and $T(a_1, b_1) \leq T(a_2, b_1) \leq T(a_2, b_2)$. The Proposition 1 implies that the boundaries of our system are the following:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Output minimum</th>
<th>Output maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = 0$</td>
<td>$\min(k_1, \frac{k_1+k_2}{2})$</td>
<td>$\max(k_1, \frac{k_1+k_2}{2})$</td>
</tr>
<tr>
<td>$b_1 = 0$</td>
<td>$\min(k_1, \frac{k_1+k_2}{2})$</td>
<td>$\max(k_1, \frac{k_1+k_2}{2})$</td>
</tr>
<tr>
<td>$a_4 = a_2$</td>
<td>$\min\left(\frac{k_2+k_3+k_1+k_4}{2}, \frac{k_1+k_2+k_3+k_4}{4}\right)$</td>
<td>$\max\left(\frac{k_2+k_3+k_1+k_4}{2}, \frac{k_1+k_2+k_3+k_4}{4}\right)$</td>
</tr>
<tr>
<td>$b_1 = b_2$</td>
<td>$\min\left(\frac{k_3+k_4}{2}, \frac{k_1+k_2+k_3+k_4}{4}\right)$</td>
<td>$\max\left(\frac{k_3+k_4}{2}, \frac{k_1+k_2+k_3+k_4}{4}\right)$</td>
</tr>
<tr>
<td>$a_2 = 1$</td>
<td>$\frac{k_3 b_1 + k_4 b_2}{b_1 + b_2} \leq \min\left(\frac{k_1+k_2}{2}\right)$</td>
<td>$\max\left(\frac{k_2 a_1 + b_1 k_1 + b_2 k_4}{a_1 + b_1 + b_2}, \frac{k_1+k_2}{2} \frac{\max(a_1, b_1) + b_1 k_1 + b_2 k_4}{\max(a_1, b_1) + b_1 + b_2}\right)$</td>
</tr>
<tr>
<td>$a_2 = 1$</td>
<td>$\min\left(\frac{k_1+k_2}{2}\right) \leq \frac{k_3 b_1 + k_4 b_2}{b_1 + b_2} \leq \max\left(\frac{k_1+k_2}{2}\right)$</td>
<td>$\min\left(\frac{k_2 a_1 + b_1 k_1 + b_2 k_4}{a_1 + b_1 + b_2}, \frac{k_1+k_2}{2} \frac{\max(a_1, b_1) + b_1 k_1 + b_2 k_4}{\max(a_1, b_1) + b_1 + b_2}\right)$</td>
</tr>
<tr>
<td>$a_2 = 1$</td>
<td>$\frac{k_3 b_1 + k_4 b_2}{b_1 + b_2} \geq \max\left(\frac{k_1+k_2}{2}\right)$</td>
<td>$\min\left(\frac{k_2 a_1 + b_1 k_1 + b_2 k_4}{a_1 + b_1 + b_2}, \frac{k_1+k_2}{2} \frac{\max(a_1, b_1) + b_1 k_1 + b_2 k_4}{\max(a_1, b_1) + b_1 + b_2}\right)$</td>
</tr>
</tbody>
</table>
Finally we have to show that these boundaries are the best possible, i.e., that for every case there exists a t-norm $T$ (a sequence of t-norms $T_n$) such that $\text{Out}_T$ (sup $\text{Out}_{T_n}$ or inf $\text{Out}_{T_n}$) is equal to the value mentioned in the table above. Since in all cases the situation is similar we will discuss just the minimum value in the case when $0 < a_1 < a_2 \leq b_1 < b_2 < 1$. Here there are again several possibilities and we will assume just $k_1 + k_2 = \min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$. Such a case corresponds to situation when $T(a_1, b_1) = 0, T(a_1, b_2) = 0$ and $T(a_2, b_1) = T(a_2, b_2) > 0$. However, $T(a_2, b_1) = T(a_2, b_2) > 0$

<table>
<thead>
<tr>
<th>Case</th>
<th>$b_2 = 1$</th>
<th>$b_2 = 1$</th>
<th>$b_2 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k_2 a_1 + k_4 a_2 \leq \min(k_1, \frac{k_1 + k_3}{2})$</td>
<td>$k_2 a_1 + k_4 a_2 \geq \max(k_1, \frac{k_1 + k_3}{2})$</td>
<td>$k_2 a_1 + k_4 a_2 \geq \max(k_1, \frac{k_1 + k_3}{2})$</td>
</tr>
<tr>
<td></td>
<td>$k_2 a_1 + k_4 a_2 \leq \min(k_1, \frac{k_1 + k_3}{2})$</td>
<td>$k_2 a_1 + k_4 a_2 \geq \max(k_1, \frac{k_1 + k_3}{2})$</td>
<td>$k_2 a_1 + k_4 a_2 \geq \max(k_1, \frac{k_1 + k_3}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\min(k_1, \frac{k_1 + k_3}{2}) \leq \frac{k_2 a_1 + k_4 a_2}{a_1 + a_2} \leq \max(k_1, \frac{k_1 + k_3}{2})$</td>
<td>$\min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
<td>$\min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
</tr>
<tr>
<td></td>
<td>$\min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
<td>$\min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
<td>$\min(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
</tr>
<tr>
<td></td>
<td>$\max(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
<td>$\max(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
<td>$\max(k_4, \frac{k_4 + k_3}{2}, \frac{k_4 + k_2}{2}, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4})$</td>
</tr>
</tbody>
</table>

Therefore, we can conclude that $T(a_1, b_1) = 0, T(a_1, b_2) = 0$ and $T(a_2, b_1) = T(a_2, b_2) > 0$.
indicate that $T$ is an ordinal sum, where one of its summands act on the interval $I$ where $[a_2, b_2] \subseteq I \subseteq [0, 1]$ and $a_1 \notin I$. Thus $T(a_1, b_1) = T(a_1, b_2) = a_1 > 0$ what is a contradiction. Therefore we need to find a sequence of t-norms $T_s$, $s \in S$ such that $\frac{k_4 + k_3}{2}$ will be equal to $\inf \text{Out}_{T_s}$. In other words we need to show that for every $\delta > 0$, small enough, there is a continuous t-norm $T_\delta$ such that $\text{Out}_{T_\delta} < k_4 + k_3 + \frac{\delta}{2}$. Note that since $k_4 + k_3 \leq k_4$ we get $k_3 \leq k_4$. Now for every small $\delta > 0$ it is enough to take a nilpotent t-norm with a piecewise linear additive generator such that $t(a_1) + t(b_1) > t(a_1) + t(b_2) \geq t(0)$, and $t(a_2) + t(b_1) = \varepsilon + t(a_2) + t(b_2)$, with $t(a_2) + t(b_1) = t(w)$, where $0 < w < a_1 - \varepsilon$ and $t(w + \varepsilon) = t(w) - \varepsilon$. Such an additive generator surely exists and we get $T(a_2, b_2) = T(a_2, b_1) + \varepsilon = w + \varepsilon$ and $T(a_1, b_1) = T(a_1, b_2) = 0$. Now $\text{Out}_T = \frac{w_k + w_k + \varepsilon}{2w + \varepsilon}$ and $\text{Out}_T - k_4 + k_3 = \frac{\varepsilon (k_4 - k_3)}{4w + 2\varepsilon}$. Therefore it is enough to start with $\varepsilon < \frac{k_4 - k_3}{2w + \varepsilon}$ and we get $\text{Out}_T - k_4 + k_3 < \delta$.

5 Case Studies

In this section we introduce two Case Studies. In the first we assume the zero-order TSK fuzzy model of the well-known Sinc function [11] and discuss the effect of isomorphism from Subsection 3.2 applied on the firing degrees. In the second Case Study we discuss the first-order TSK fuzzy model in the well-know Noise Pollution problem [8].

5.1 Sinc function and isomorphism

In the following Case Study we show the effect of the isomorphism $g$ applied to the product and the Lukasiewicz t-norm. We assume the zero-order TSK fuzzy model of the Sinc function given for $(x, y) \in [-10, 10]^2$ by $\text{Sinc}(x, y) = \frac{\sin(x) \sin(y)}{xy}$.

In order to model the Sinc function we define 5 triangular fuzzy sets for every input with centers $C_i = -10 + \frac{i10}{5}$ for $i \in \{1, 2, 3, 4, 5\}$, with the width $\frac{10}{5}$. Together we get 25 rules and we see that there can be maximally 4 rules active for one input vector $(x, y)$. The consequence constant for each rule corresponds to the value of the Sinc function in the cores
of two fuzzy sets corresponding to the given rule. In the next step we would like to improve performance of our system by using the isomorphism \( g \). We will restrict our selves in this case to power transformations and piecewise linear transformations, i.e., to isomorphisms of the form \( g(x) = x^p \) for \( p \in ]0, \infty[ \) and \( g(x) = \max(\frac{p}{q}x, \frac{(1-p)x+p-q}{1-q}) \), \( p, q \in ]0, 1[ \). Note that t-norm \( T_P \) is invariant with respect to power transformations and thus the choice of appropriate power \( p \) influence only the shape of the fuzzy sets in the equivalent zero order TSK system. In the case of \( T_L \) the equivalent zero order TSK system has fuzzy sets which are transformed in the same way as in the case of \( T_P \) but here the power transformation changes the t-norm \( T_L \) to the t-norm \( T_{SS}^p \), i.e., to the Schweizer-Sklar t-norm with the parameter \( p \).

The testing data were taken from [11]. The error of the system is computed by

\[
\text{Error} = \frac{1}{121} \sum_{i=1}^{11} \sum_{j=1}^{11} |\text{Out}_T(x(i), y(j)) - \text{Sinc}(x(i), y(j))|.
\]

The results for both types of isomorphisms can be found in the following tables. Note that the percentages indicate the percentage difference between the error of the system when isomorphism was used and the error of the basic system, i.e., it is \( \frac{\text{Err}_g - \text{Err}}{\text{Err}} \cdot 100\% \).

<table>
<thead>
<tr>
<th>Power transformation</th>
<th>Piece-wise linear transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p ) ( T_L )</td>
<td>( T_P )</td>
</tr>
<tr>
<td>1 0.0397 0.0338</td>
<td>8.149% 16.761%</td>
</tr>
<tr>
<td>2 2.035% 4.365%</td>
<td>20.739% 39.053%</td>
</tr>
<tr>
<td>0.5 0.1 0.2 -7.35% -9.82%</td>
<td></td>
</tr>
<tr>
<td>4 -2.035% -4.365%</td>
<td>0.3 0.8 -6.18% -8.02%</td>
</tr>
<tr>
<td>0.25 -4.255% -9.432%</td>
<td>0.4 0.9 -6.81% -8.02%</td>
</tr>
<tr>
<td>6 28.161% 50.049%</td>
<td>0.5 0.8 -2.85% -3.36%</td>
</tr>
<tr>
<td>0.1667 -4.916% -10.922%</td>
<td>0.7 0.2 0.00% 0.00%</td>
</tr>
<tr>
<td>0.125 -4.418% -10.363%</td>
<td>0.9 0.1 0.00% 0.00%</td>
</tr>
</tbody>
</table>

As we can see from these results – in the case of Sinc function – the class of strict t-norms
is a better choice. We also see that the best result in the case of power transformation was obtained for \( p = 0.1667 \) (10.922\% decrease in error). In the case of piecewise linear transformation the best result was obtained for \( p = 0.1 \) and \( q = 0.7 \) (15.9\% decrease in error). Note that for the minimum t-norm the error of the system is 0.0316 \( \left( \frac{\text{Err}_{M}}{\text{Err}_{TP}} \right) \cdot 100\% = -6.29\% \). During the data analysis one can see that for some inputs, the Sinc function gives the values which cannot be obtained for any continuous t-norm. Indeed, if we use the output boundaries from Section 4 we can see for which inputs the Sinc function gives the values that are out of these boundaries. It is clear that for such inputs we cannot get a better fit than the upper or the lower boundary. If we compute together the differences (between the Sinc function and a closer boundary value) for all such inputs and divide the sum by 121 we get the value of the minimal possible error. In our case this minimal value is 0.0237. We have seen that the use of the isomorphism \( g \) can decrease the initial error by more than 15\%. However, it always depends on the type of application whether such decrease is significant or not.

### 5.2 Noise Pollution and fuzzy control

Noise pollution in major conurbations leads to a general degradation of quality of life. Fichera and colleagues [8] have used conventional models, Takagi-Sugeno-Kang fuzzy models, and neuro-fuzzy models to simulate noise pollution in a small town in Italy. They have found that the noise level can be predicted using:

(i) The number of vehicles per hour was computed as an aggregate of a variety of vehicles (cars, motor bikes, and heavy goods vehicles) per hour:

\[
neq = ncars + 3 \cdot n_{motor \ bikes} + 6 \cdot n_{heavy \ goods \ vehicles}.
\]

(ii) The so-called equivalent continuous noise level \( (L_{AceqT}) \) is defined as a function of the number of vehicles per hour \( (neq) \), the average height \( h \), of the buildings along a road
that has an average width $w$:

$$L_{AeqT} = f(n_{eq}, h, w).$$

Fichera et al. defined the term set $n_{eq}$ as comprising the linguistic variables small and large. Here $\mu_{\text{small}}$ and $\mu_{\text{large}}$ are piecewise linear membership functions with $\mu_{\text{small}}(x) = 1$ for $x \leq 923$, $\mu_{\text{small}}(x) = 0$ for $x \geq 10489$ and $\mu_{\text{large}}(x) = 0$ for $x \leq 924$, $\mu_{\text{large}}(x) = 1$ for $x \geq 8944$.

The height of the building term set $h$ comprises tall and low buildings. Here again $\mu_{\text{tall}}$ and $\mu_{\text{low}}$ are piecewise linear membership functions with $\mu_{\text{tall}}(x) = 0$ for $x \leq 12.44$, $\mu_{\text{tall}}(x) = 1$ for $x \geq 34.49$ and $\mu_{\text{low}}(x) = 1$ for $x \leq 12.7$, $\mu_{\text{low}}(x) = 0$ for $x \geq 31.88$.

### 5.2.1 Noise Pollution – Rule Base

The fuzzy rule set for computing the noise level is a revised and perhaps corrected version of the original in Fichera et al.:

- If $n_{eq}$ is small AND $h$ is low Then $L_{eq} = -148.6 - 0.087n_{eq} + 24.38h + 0.24w$
- If $n_{eq}$ is small AND $h$ is low Then $L_{eq} = -894.2 + 0.087n_{eq} + 26.53h - 0.09w$
- If $n_{eq}$ is small AND $h$ is low Then $L_{eq} = 1180 - 0.071n_{eq} - 26.99h - 0.61w$
- If $n_{eq}$ is small AND $h$ is low Then $L_{eq} = 413.99 + 0.072n_{eq} - 30.97h - 1.99w$

An example given by Fichera et al. was the situation where $n_{eq} = 5000$, $h = 15$ and $w = 30$. In the following we will try to produce output of this system for different t-norms used as AND operator. Note that in Fichera et al. the minimum t-norm $T_M$ was used.

The outputs of all four rules for $n_{eq} = 5000$, $h = 15$ and $w = 30$ are given in the following table
<table>
<thead>
<tr>
<th>Rule</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-210.7</td>
<td>-63.95</td>
<td>401.85</td>
<td>249.74</td>
<td></td>
</tr>
</tbody>
</table>

Membership degrees for \( n_{eq} = 5000 \) and \( h = 15 \) are \( \mu_{small}(5000) = 0.574, \mu_{large}(5000) = 0.508, \mu_{low}(x) = 0.88, \mu_{tall}(x) = 0.116. \)

### 5.2.2 Noise Pollution – Main Results

For t-norms \( T_M, T_P, T_L \) we get

<table>
<thead>
<tr>
<th>t-norm</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \tau_3 )</th>
<th>( \tau_4 )</th>
<th>OUTPUT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_M )</td>
<td>0.574</td>
<td>0.116</td>
<td>0.508</td>
<td>0.116</td>
<td>79.82</td>
</tr>
<tr>
<td>( T_P )</td>
<td>0.505</td>
<td>0.067</td>
<td>0.447</td>
<td>0.059</td>
<td>77.757</td>
</tr>
<tr>
<td>( T_L )</td>
<td>0.454</td>
<td>0.0</td>
<td>0.388</td>
<td>0.0</td>
<td>71.728</td>
</tr>
</tbody>
</table>

where \( \tau_i \) is a firing degree of \( i \)-th rule.

Although already between basic t-norms we can see the big differences, there are t-norms for which the output is even much more different. For example for Schweizer-Sklar t-norm with \( \lambda = 2 \) given by \( T^2(x, y) = \max(0, x^2 + y^2 - 1) \frac{1}{2} \) the firing degrees \( \tau_i \) of these four rules are \( 0.322, 0.181, 0.0, 0.0 \), respectively and output of the system is 9.822.

From Section 4 we see that in this case \( a_1 = 0.508, a_2 = 0.574 \) and \( b_1 = 0.116, b_2 = 0.88 \) with \( k_1 = 249.74, k_2 = 401.85, k_3 = -63.95, k_4 = -210.7 \). Therefore \( 0 < b_1 < a_1 < a_2 < b_2 < 1 \) and maximal possible output is \( \max(k_4, k_4 + k_3, \frac{k_4 + k_3 + k_2}{3}, \frac{k_4 + k_3 + k_2 + k_1}{4}) = \max(-210.7, \frac{-210.7 + 401.85}{2}, \frac{-210.7 + 401.85 - 63.95}{3}, \frac{-210.7 + 401.85 - 63.95 + 249.74}{4}) = \max(-210.7, 95.575, 42.4, 94.235) = 95.575. \) Similarly, minimal possible output is \( \min(-210.7, 95.575, 42.4, 94.235) = -210.7. \)

Summarizing the results we see that if we start with \( T_M \) as a reference point, for arbitrary continuous t-norm \( T \) we have \( \frac{Out_T - Out_{TM}}{Out_{TM}} \cdot 100\% \in [-364\%, 19.7\%] \).
6 Afterword and Future Work

There are key differences in the output of fuzzy control systems when the variables do not have the idiosyncratic values (for 0 and 1 we have $T(0, x) = 0$ and $T(1, x) = x$ for all t-norms $T$). This we have demonstrated in our case studies (and by formulas in Section 2). In Section 2 we have studied a simple zero-order TSK system which employs different t-norms. We have shown, that the result is always a convex combination of the consequent constants (what follows from the definition of a TSK system) and thus, the output of the system with two rules employed can always be computed as a linear transformation of a system with consequent constants $-1, 1$. Since effect of the different t-norms on a simple TSK system, with consequent constants $-1, 1$ can be expressed numerically (by integration) we are able to compare this effect for any general simple TSK system (with consequence constants $k_1, k_2$).

We have further shown, that the choice of a continuous t-norm as AND operator can be limited to the class of strict (nilpotent) t-norms only. Further, we have shown that it is enough to assume just the product t-norm (Łukasiewicz t-norm) and take the isomorphic transformation of firing degrees $g$. We have also derived the output boundaries for zero-order TSK system. Finally, we have applied our results in the Case Studies.

In this paper we have studied fuzzy controllers where at any given time only two (four) rules were operational. What happens when there are many rules? Or, perhaps, more importantly in neuro-fuzzy system where the small error in the output will matter during the training of the neuro-fuzzy system. The computation of errors in the forward and backward passes during the training of an adaptive neuro-fuzzy system perhaps will be affected by the choice of t-norms. To investigate this possible build-up of error we plan to implement a system similar to that proposed by Jang, Sun and Mitzuani [12]. Our system will be based on their ANFIS algorithm for delineating the effects of the choice of a (continuous) triangular norm.
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References


Appendix

Proposition 1

**Proof:** Assume a non-increasing permutation $(\cdot) : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\}$, i.e., $c(1) \geq c(2) \geq c(3) \geq c(4)$. Then we have $c(1) + c(2) + c(3) + c(4) = 1$. 

We want to find the best boundaries for

\[ D = c(1)k(1) + c(2)k(2) + c(3)k(3) + c(4)k(4), \]  

where \( k(1), k(2), k(3), k(4) \in \mathbb{R}. \)

First we will find maximum of \( D. \) There are the following possibilities:

(i) If \( k(1) \geq k(2). \) Here since \((k(1) - k(2))(c(2) - c(3)) \geq 0\) we have

\[ c(1)k(1) + c(2)k(2) + c(3)k(3) + c(4)k(4) \leq \]

\[ k(1)(c(1) + c(2) - c(3)) + k(2)c(3) + k(3)c(3) + k(4)c(4). \]

Now there are two possibilities:

(a) \( 2k(1) \geq k(2) + k(3) \) : in this case the inequality \((2k(1) - k(2) - k(3))(c(3) - c(4)) \geq 0\) implies

\[ k(1)(c(1) + c(2) - c(3)) + k(2)c(3) + k(3)c(3) + k(4)c(4) \leq \]

\[ k(1)(c(1) + c(2) + c(3) - 2c(4)) + k(2)c(4) + k(3)c(4) + k(4)c(4). \]

Since \( c(1) + c(2) + c(3) = 1 - c(4) \) the last expression is equal to

\[ k(1)(c(1) + c(2) + c(3) - 2c(4)) + k(2)c(4) + k(3)c(4) + k(4)c(4) = \]

\[ k(1)(1 - 3c(4)) + (k(2) + k(3) + k(4))c(4) = \]

\[ k(1) + c(4)(k(2) + k(3) + k(4) - 3k(1)) \]

and \( 1 \geq 1 - 3c(4) \geq c(4) \geq 0. \) Therefore \( \frac{1}{4} \geq c(4) \geq 0 \) and for \( MAX = \)
\[
\max_{\frac{1}{4} \geq c(4) \geq 0} k(1)(1 - 3c(4)) + (k(2) + k(3) + k(4))c(4)
\]
we have
\[
MAX = \begin{cases} 
\frac{k(1)+k(2)+k(3)+k(4)}{4} & \text{if } k(2) + k(3) + k(4) - 3k(1) \geq 0 \\
k(1) & \text{else.}
\end{cases}
\]

(b) \(2k(1) < k(2) + k(3)\) : in this case the inequality \((k(2) + k(3) - 2k(1))(\frac{c(1)+c(2)+2c(3)}{3}) \geq 0\) implies
\[
k(1)(c(1) + c(2) - c(3)) + k(2)c(3) + k(3)c(3) + k(4)c(4) \leq
(k(1) + k(2) + k(3))\frac{c(1) + c(2) + c(3)}{3} + k(4)c(4).
\]
Since \(c(1) + c(2) + c(3) = 1 - c(4)\) the last expression is equal to
\[
(k(1) + k(2) + k(3))\frac{c(1) + c(2) + c(3)}{3} + k(4)c(4) =
(k(1) + k(2) + k(3))\frac{1 - c(4)}{3} + k(4)c(4) =
\frac{k(1) + k(2) + k(3)}{3} + c(4)(k(4) - \frac{k(1) + k(2) + k(3)}{3})
\]
and \(1 \geq \frac{1-c(4)}{3} \geq c(4) \geq 0\). Therefore \(\frac{1}{4} \geq c(4) \geq 0\) and if we denote \(MAX = \max_{\frac{1}{4} \geq c(4) \geq 0} \frac{k(1)+k(2)+k(3)+k(4)}{3} + c(4)(k(4) - \frac{k(1)+k(2)+k(3)}{3})\) we have
\[
MAX = \begin{cases} 
\frac{k(1)+k(2)+k(3)+k(4)}{4} & \text{if } k(1) + k(2) + k(3) - 3k(4) \leq 0 \\
\frac{k(1)+k(2)+k(3)}{3} & \text{else.}
\end{cases}
\]

(ii) If \(k(1) < k(2)\). Here since \((k(2) - k(1))\frac{c(1)-c(2)}{2} \geq 0\) we have
\[
c(1)k(1)+c(2)k(2)+c(3)k(3)+c(4)k(4) \leq (k(1)+k(2))(\frac{c(1)+c(2)}{2})+k(3)c(3)+k(4)c(4).
\]
Now we have the following possibilities:

(a) \( k(1) + k(2) \geq 2k(3) \): in this case the inequality \( (c(3) - c(4))\left(\frac{k(1)+k(2)}{2} - k(3)\right) \geq 0 \) implies

\[
(k(1) + k(2))\left(\frac{c(1) + c(2)}{2}\right) + k(3)c(3) + k(4)c(4) \leq \\
(k(1) + k(2))\left(\frac{\frac{c(1) + c(2)}{2} + c(3) - c(4)}{2}\right) + (k(3) + k(4))c(4).
\]

Since \( c(1) + c(2) + c(3) = 1 - c(4) \) the last expression is equal to

\[
(k(1) + k(2))\left(\frac{c(1) + c(2) + c(3) - c(4)}{2}\right) + (k(3) + k(4))c(4) = \\
(k(1) + k(2))\left(\frac{1 - 2c(4)}{2}\right) + (k(3) + k(4))c(4) = \\
\frac{k(1) + k(2)}{2} + c(4)(k(3) + k(4) - k(1) - k(2))
\]

and \( 1 \geq \frac{1 - 2c(4)}{2} \geq c(4) \geq 0 \). Therefore \( \frac{1}{4} \geq c(4) \geq 0 \) and if we denote \( MAX = \max_{\frac{1}{4} \geq c(4) \geq 0} \frac{k(1)+k(2)+k(3)+k(4)}{4} + c(4)(k(3) + k(4) - k(1) - k(2)) \) we have

\[
MAX = \begin{cases} 
\frac{k(1)+k(2)+k(3)+k(4)}{4} & \text{if } k(3) + k(4) \geq k(1) + k(2) \\
\frac{k(1)+k(2)}{2} & \text{else.}
\end{cases}
\]

(b) \( k(1) + k(2) < 2k(3) \): in this case the inequality \( (k(3) - \frac{k(1)+k(2)}{2})\left(\frac{c(1)+c(2)-2c(3)}{3}\right) \) implies

\[
(k(1) + k(2))\left(\frac{c(1) + c(2)}{2}\right) + k(3)c(3) + k(4)c(4) \leq \\
(k(1) + k(2) + k(3))\left(\frac{\frac{c(1) + c(2)}{2} + c(3)}{3}\right) + k(4)c(4).
\]
Since \( c(1) + c(2) + c(3) = 1 - c(4) \) the last expression is equal to

\[
(k(1) + k(2) + k(3)) \left( \frac{c(1) + c(2) + c(3)}{3} \right) + k(4)c(4) =
\]

\[
\frac{k(1) + k(2) + k(3)}{3} + c(4)k(4) - \frac{k(1) + k(2) + k(3)}{3}
\]

and \( 1 \geq \frac{1-c(4)}{3} \geq c(4) \geq 0 \). Thus \( \frac{1}{4} \geq c \geq 0 \) and for \( \text{MAX} = \max_{\frac{1}{4} \geq c(4) \geq 0} k(1) + k(2) + k(3) + k(4) \) we have

\[
\text{MAX} = \begin{cases} 
\frac{k(1) + k(2) + k(3) + k(4)}{4} & \text{if } k(1) + k(2) + k(3) \leq 3k(4) \\
\frac{k(1) + k(2) + k(3)}{3} & \text{else}.
\end{cases}
\]

Summarizing

\[
c(1)k(1) + c(2)k(2) + c(3)k(3) + c(4)k(4) \leq \\
\max(k(1), \frac{k(1) + k(2)}{2}, \frac{k(1) + k(2) + k(3)}{3}, \frac{k(1) + k(2) + k(3) + k(4)}{4}).
\]

The proof for minimum is analogous. Here

\[
c(1)k(1) + c(2)k(2) + c(3)k(3) + c(4)k(4) \geq \\
\min(k(1), \frac{k(1) + k(2)}{2}, \frac{k(1) + k(2) + k(3)}{3}, \frac{k(1) + k(2) + k(3) + k(4)}{4}).
\]

\[\square\]