Stochastic Subgradient Methods with Approximate Lagrange Multipliers

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***Abstract*—We study the use of approximate Lagrange multipliers in the stochastic subgradient method for the dual problem in convex optimisation. The use of approximate Lagrange multipliers in the optimisation (instead of the true multipliers) is motivated by the fact that we can cast some typically thought non-convex control problems as convex optimisations. For example, we can solve some stochastic discrete decision making problems by simply solving a sequence of unconstrained convex optimisations. The use of approximate multipliers allows us also to solve problems where there are constraints on the order in which control actions can be made.**

***Index Terms*—convex optimisation, max-weight scheduling, subgradient methods, approximate multipliers, stochastic control.**

I. Introduction

In this paper we study the use of approximate Lagrange multipliers in subgradient methods for constrained convex optimisation problems. One of the motivations of using an approximation of the Lagrange multipliers is that in some problems the exact multipliers might not be available, but a "perturbed" or "outdated" version of the multipliers is available instead. For example, in distributed optimisation the exchange of system state information might be affected by transmission delays or losses. Another motivation of using approximate Lagrange multipliers is because of its applications to network scheduling problems involving queues.

As shown in [1], it is possible to identify scaled queues occupancies with approximate Lagrange multipliers, *i.e.*, the Lagrange multiplier λ_k generated by the subgradient method stays close to the α-scaled queue occupancy αQ_k in a network (where α is the step size used in the subgradient method). A direct consequence of this is that the scaled queues occupancies in networks can be used as surrogate quantities of the Lagrange multipliers in the subgradient method for the dual problem. In other words, by using approximate multipliers it is possible to cast some discrete decision making problems as convex optimisations.

Our convex approach to solving network/scheduling optimisation problems is in marked contrast to approaches like max-weight [2] that are built on Lyapunov optimisation theory. Whereas max-weight approaches focus on constructing a policy that stabilises a system, we instead solve the dual problem and then derive the stability of the system via the properties of the dual problem. This way of looking at the problem allows us to construct scheduling policies in a more flexible manner than with max-weight approaches. In particular, we can design scheduling policies that have constraints on the order in which discrete actions can be selected.

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A. Related Work

The max-weight scheduling algorithm was proposed by Tassiulas and Ephremides in [2]. The algorithm is able to find an optimal scheduling policy (which consists of choosing the discrete action that minimises the sum of the squared queues backlogs) without previous knowledge of the mean packet arrival rates in the system. The max-weight algorithm has been extended in a sequel of papers [3], [4], [5] to consider the minimisation of a convex utility function, and to general functions in [6]. The works in [5], [6] also extend max-weight to allow convex constraints. The construction of max-weight policies with constraints (or penalties) is proposed in [7]. There the authors consider the problem of dynamic allocation of servers with switchover delay.

The work in [8] studies the allocation of rates in a network while providing fairness using a dynamical system approach and Lyapunov stability. In [9] the authors use a convex approach and study the optimal allocation of rates using subgradient methods. Importantly, [8] and [9] work with continuous valued variables rather than with the discrete actions used in the max-weight approaches [2], [3], [4], [5], [6]. The works in [10], [11] are the first ones that establish rigorous connections between the max-weight and dual approaches in convex optimisation via Lagrange multipliers.

II. Convex Optimisation

Consider the following convex optimisation problem P in standard form:

$$\underset{x\in X}{\text{minimise}} \quad f(x) \tag{1}$$

$$\text{subject to} \quad g_i(x) \leq 0 \qquad i = 1,\dots,m \tag{2}$$

where $f, g_i : X \to \mathbb{R}$ are convex functions and $X \subseteq \mathbb{R}^n$ a convex set. We will always assume that set $X_0 := \{x \in X \mid g_i(x) \leq 0, i = 1,\dots,m\} \neq \emptyset$, and so problem P is feasible. Further, using standard notation we let $f^\star := \min_{x\in X_0} f(x)$ and $x^\star \in \arg\min_{x\in X_0} f(x)$.

As it is usual in convex optimisation, the Lagrange dual function of problem P is given by

$$q(\lambda) = \inf_{x\in X} L(x,\lambda) = \inf_{x\in X}\{f(x) + \sum_{i=1}^{m} \lambda^{(i)} g_i(x)\}, \tag{3}$$

where $\lambda^{(j)}$ is the j'th component of vector $\lambda \in \mathbb{R}^m_+$. Recall that since $q(\lambda)$ is the infimum of a set of linear functions it is concave. In order to keep notation short we will write $L(x, \lambda) = f(x) + \lambda^T g(x)$ where $g = [g_1, \ldots, g_m]^T$, and define $X(\lambda) := \{\arg\min_{x \in X} L(x, \lambda)\}$ to be the set of minima of $L(\cdot, \lambda)$ for a given vector λ.

The following two assumptions are key in our work.

Assumption 1 (Bounded Set)**.** *X is convex and bounded.*

Assumption 2 (Slater Condition)**.** $\text{relint}(X_0)$ *is non-empty, i.e., there exists a point $x \in X$ such that $g(x) \prec 0$.*

Assumption 1 is important because it guarantees that q is Lipschitz continuous with constant $L_g := \max_{x \in X} \|g(x)\|_2$, and Assumption 2 ensures that strong duality holds. Namely, the solution of the dual problem P^D,

$$\underset{\lambda \succeq 0}{\text{maximise}} \quad q(\lambda) \tag{4}$$

coincides with the solution of the primal problem P. That is, $\max_{\lambda \succeq 0} q(\lambda) =: q(\lambda^\star) = f^\star$ where $\lambda^\star \in \arg\max_{\lambda \succeq 0} q(\lambda)$. Another consequence of the Slater condition is that the set of dual optima, $\Lambda^\star := \{\arg\max_{\lambda \succeq 0} q(\lambda)\}$, is a bounded subset from $\mathbb{R}^m_+$ (see Lemma 2 in [12]).

A. Subgradient Method for the Dual Problem

Problem P^D is an unconstrained concave optimisation problem that can be solved using the subgradient method. Recall that when the dual function is differentiable then the subgradient is indeed the gradient. In short, the subgradient method consists of the following update

$$\lambda_{k+1} = [\lambda_k + \alpha g(x_k)]^+, \tag{5}$$

where $[\cdot]^+ = \max\{0, \cdot\}$, $\lambda_1 \in \mathbb{R}^m$, $g(x_k)$ is the subgradient of q at point λ_k, and $\alpha > 0$ is a constant step size. The subgradient method can make use of more complex step sizes, but as we will show in Section II-B, constant step size will play a prominent role when working with averages.

Lemma 1 (Subgradient Method)**.** *Consider optimisation problem P^D and the subgradient update (5) where $x_k \in X(\lambda_k)$ and $\alpha > 0$. Suppose Assumptions 1 and 2 hold. Then, there exists a non-negative and non-increasing sequence $\{\beta_k\}$ that converges to $\alpha(3/2)L_g^2$ for k large enough and $\beta_k \leq q(\lambda_k) - q(\lambda^\star) \leq 0$ for all k.*

Proof: See the appendix. ■

The intuition behind the lemma is that λ_k is monotonically attracted to a vector $\lambda^\star \in \Lambda^\star$ until it converges to a ball around $\Lambda^\star$, the size of which depends on parameter α. Hence, by the Lipchitz continuity of the dual function we can then expect that $q(\lambda_k)$ converges (probably not monotonically) to a ball around $q(\lambda^\star)$. Moreover, since λ_k is monotonically attracted to a ball around $\Lambda^\star$, and $\Lambda^\star$ is a bounded set from $\mathbb{R}^m$ from Assumption 2, it follows that λ_k is bounded for all k. See [12, Lemma 3] for a bound on λ_k. Figure 1 illustrates schematically the convergence of $q(\lambda_k)$ to level set $q(\lambda^\star) - q(\lambda_k) \leq \alpha(3/2)L_g^2$.

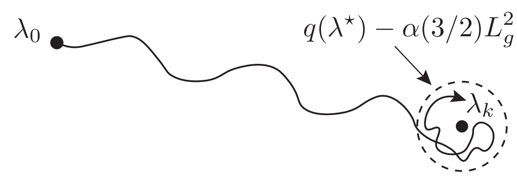

Fig. 1. Illustrating the convergence of the subgradient method for the dual problem with constant step size.

B. Approximate Primal Solutions Using Averages

We now proceed to show we can recover approximate primal solutions using the subgradient method for the dual problem. First we present the following lemma which relates bounded Lagrange multipliers and the feasibility of averages of primal variables.

Lemma 2 (Feasible Solutions using Averages)**.** *Let $\{x_k\}$ be a sequence of points in X and consider update $\lambda_{k+1} = [\lambda_k + \alpha g(x_k)]^+$. Then, $g(\bar{x}_k) \preceq \lambda_{k+1}/k$, where $\bar{x}_k = \frac{1}{k}\sum_{i=1}^k x_i$.*

Proof: See the appendix. ■

Observe from Lemma 2 that if λ_{k+1} is bounded for all k (which is the case in the subgradient method) then $\bar{x}_k$ is attracted to a feasible point as $k \to \infty$. We now present one of our main results, which is a refinement of Theorem 3 in [13] with sharper bounds and simpler proof.

Theorem 1 (Approximate Primal Solutions using Averages)**.** *Consider the subgradient method for the dual problem with constant step size $\alpha > 0$,* i.e.,

$$\lambda_{k+1} = [\lambda_k + \alpha g(x_k)]^+, \tag{6}$$

where $x_k \in X(\lambda_k) := \{\arg\min_{x \in X} L(x, \lambda_k)\}$. Suppose Assumptions 1 and 2 hold. Then,

$$-\beta_k - \frac{\lambda_k^\circ}{\alpha k} \leq f(\bar{x}_k) - f^\star \leq \frac{\alpha L_g^2}{2} + \frac{\|\lambda_1\|_2^2}{2\alpha k}, \tag{7}$$

where $\lambda_k^\circ = \lambda_k^T \lambda_{k+1}$, $\bar{x}_k := \frac{1}{k}\sum_{i=1}^k x_i$ and $\{\beta_k\}$ is a sequence given by the subgradient method.

Proof: See the appendix. ■

Theorem 1 says that $f(\bar{x}_k)$ converges to a solution close to $f^\star$ for k sufficiently large and α sufficiently small. Observe that if we let $k \to \infty$ we obtain

$$\frac{\alpha 3 L_g^2}{2} \leq f(\bar{x}_k) - f^\star \leq \frac{\alpha L_g^2}{2}, \tag{8}$$

and if we make $\alpha \to 0$ then $f(\bar{x}_k) \to f^\star$. Further, since by Lemma 2 we have that $g(\bar{x}_k) \preceq 0$ when $k \to \infty$, we can conclude that $\bar{x}_k \to x^\star$. How fast sequence $\{\beta_k\}$ converges to $\alpha(3/2)L_g^2$ will depend on the step size and characteristics of the objective function and constraints. For example, if $q(\lambda)$ were strongly convex we could use second order methods and obtain a sequence $\{\beta_k\}$ that converges fast to $\alpha(3/2)L_g^2$.

Theorem 1 can be extended to consider the case where the subgradient $g(x_k)$ is computed using a point nearby to λ_k. Formally, $x_k \in X(\mu_k)$ where μ_k is an "approximate" Lagrange multiplier of λ_k that satisfies $\|\lambda_k - \mu_k\|_2 \leq \epsilon/L_g$ for all k and $\epsilon \geq 0$. As we will show in Section III it

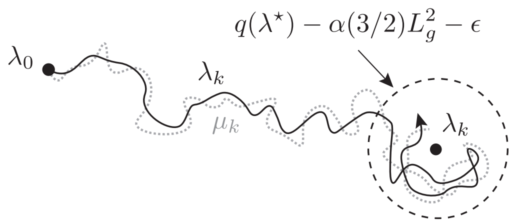

Fig. 2. Illustrating the convergence of the subgradient method for the dual problem with constant step size when the subgradients are obtained using nearby (dual) points or approximate Lagrange multipliers. Observe from the figure that the approximate Lagrange multiplier μ_k (dashed line) is always close to the Lagrange multiplier λ_k (straight line).

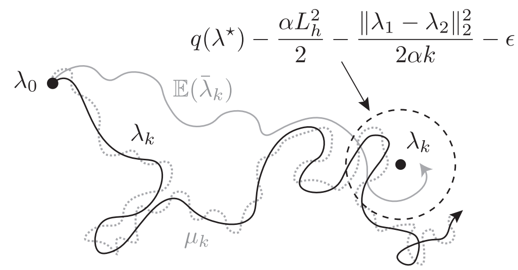

Fig. 3. Illustrating the convergence of the stochastic subgradient method with approximate Lagrange multipliers. In contrast to the deterministic case (see Figure 2), we now have that $\bar{\lambda}_k$ converges to a ball around $\lambda^\star$.

will be possible to identify scaled queues occupancies with approximate multipliers μ_k. A consequence of this is that scaled queues occupancies can be used as surrogate quantities in network optimisation. We have the following corollary to Theorem 1.

Corollary 1. *Consider the setup of Theorem 1 where update (6) now uses $x_k \in X(\mu_k)$ where $\|\lambda_k - \mu_k\|_2 \leq \epsilon/L_g$ for all $k = 1, 2, \dots$ and $\epsilon \geq 0$. Then, the bound (7) holds and sequence $\{\beta_k\}$ converges to $\alpha(3/2)L_g^2 + \epsilon$ for k sufficiently large.*

The use of a nearby or approximate Lagrange multiplier to compute the subgradient of the dual function is in fact equivalent to the subgradient method for the dual problem with ϵ-subgradients – see [14, pp. 625]). Observe that $|q(\lambda_k) - q(\mu_k)| \leq \|\lambda_k - \mu_k\|_2 L_g \leq \epsilon$. Figure 2 illustrates the convergence of $q(\lambda_k)$ to a ball around $q(\lambda^\star)$ when the subgradient method is computed using a nearby Lagrange multiplier. Compare Figure 1 and 2 to see that the ball to which $q(\lambda_k)$ converges to depends now on parameter ϵ (which controls the distance between λ_k and μ_k).

C. Subgradient Method with Noisy Dual Updates

The subgradient method for the dual problem can be extended to consider noisy dual updates. We have the following lemma.

Lemma 3 (Stochastic Subgradient Method with Approximate Lagrange Multipliers)**.** *Consider optimisation problem P^D and the subgradient update*

$$\lambda_{k+1} = [\lambda_k + \alpha(g(x_k) + B_k)]^+, \tag{9}$$

where $x_k \in X(\mu_k)$ with $\|\lambda_k - \mu_k\|_2 \leq \epsilon/L_g$, $\epsilon \geq 0$ and B_k are i.i.d. random variables with mean $b \in \mathbb{R}^m$ and finite variance σ_B. Suppose Assumption 1 holds and that there exists a point $x \in X$ such that $g(x) + b \prec 0$. Also, suppose the subgradients of the dual function are bounded for all k, i.e.*, $\max_{x \in X} \|g(x) + B_k\|_2 \leq L_h$ for all k. Then,*

$$-\frac{\|\lambda_1 - \lambda^\star\|_2^2}{2\alpha k} - \frac{\alpha L_h^2}{2} - \epsilon \leq q(\mathbb{E}(\bar{\lambda}_k)) - q(\lambda^\star) \leq 0, \tag{10}$$

where $\bar{\lambda}_k := \frac{1}{k}\sum_{i=1}^k \lambda_i$.

Proof: See the appendix. ■

In the stochastic subgradient method we have that the *expected value* of the running average $\bar{\lambda}_k$ is attracted to a ball around $\Lambda^\star$. The latter is schematically illustrated in Figure 3. An important point to note is that now we require that the Slater condition holds with the mean of the random variable, *i.e.*, there exists a point $x \in X$ such that $g(x) + b \prec 0$ and not just $g(x) \prec 0$.

We can obtain approximate primal solutions using the stochastic subgradient update.

Theorem 2 (Approximate Primal Solutions using Averages, Approximate Lagrange Multipliers and Noisy Dual Updates)**.** *Consider the stochastic subgradient method of Lemma 3. Suppose Assumption 1 holds and that there exists a point $x \in X$ such that $g(x) + b \prec 0$. Also, suppose the subgradients of the dual function are bounded for all k,* i.e.*, $\max_{x \in X} \|g(x) + B_k\|_2 \leq L_h$ for all k. Then,*

$$-\frac{\|\lambda_1 - \lambda^\star\|_2^2}{2\alpha k} - \frac{\alpha L_h^2}{2} - \frac{\lambda_k^\circ}{\alpha k} - \epsilon \tag{11}$$

$$\leq f(\mathbb{E}(\bar{x}_k)) - f^\star \leq \frac{\alpha L_h^2}{2} + \frac{\|\lambda_1\|_2^2}{2\alpha k}, \tag{12}$$

where $\bar{x}_k := \frac{1}{k}\sum_{i=1}^k x_i$ and $\lambda_k^\circ = \mathbb{E}[\bar{\lambda}_k]^T \mathbb{E}[\lambda_{k+1}]$.

Proof: See the appendix. ■

Theorem 2 says that $f(\mathbb{E}(\bar{x}_k))$ converges to a ball around $f^\star$ for k sufficiently large, and when $k \to \infty$ we have that

$$-\frac{\alpha L_h^2}{2} - \epsilon \leq f(\mathbb{E}(\bar{x}_k)) - f^\star \leq \frac{\alpha L_h^2}{2}.$$

Further, if $\alpha \to 0$ then $f(\mathbb{E}(\bar{x}_k)) \to f^\star$ as $k \to \infty$.

III. Approximate Lagrange Multipliers & Queue Continuity

Queues updates in networks are usually given by

$$Q_{k+1} = [Q_k + \delta_k]^+,$$

where $Q_1 \in \mathbb{R}^m$ and $\delta_k \in \mathbb{Z}^m$. Difference δ_k is discrete and represents the change of discrete quantities (packets, cars, people, etc.) that get in/out of the queue in a time slot k. Next, observe that since $[\cdot]^+ = \max\{0, \cdot\}$ is a homogeneous function, if we let $\alpha Q_k := \mu_k$ we can write

$$\mu_{k+1} = [\mu_k + \alpha\delta_k]^+,$$

where α is the step size used in the subgradient method. That is, an α-scaled queue is like a subgradient update with the difference that δ_k is constrained to be discrete. Recall that μ_k will be an approximate Lagrange multiplier if it stays uniformly close to λ_k for all k. This will be actually the case when the sequence of discrete quantities $\{\delta_k\}$ and subgradients $\{g(x_k)\}$ stay "close" in appropriate sense, and constraints in the optimisation P are linear [13]. We have the following lemma, which is a restatement of [15, Proposition 3.1.2].

Lemma 4 (Continuity of the Skorokhod Map)**.** *Consider updates* $\lambda_{k+1} = [\lambda_k + \alpha x_k]^+, \mu_{k+1} = [\mu_k + \alpha y_k]^+$ *where* $\lambda_1 = \mu_1 \geq 0$*,* $\alpha > 0$*, and* $\{x_k\}$ *and* $\{y_k\}$ *are two sequences of points from* $\mathbb{R}$ *such that* $|\sum_{i=1}^k (x_i - y_i)| \leq \epsilon$*. Then,*

$$|\lambda_{k+1} - \mu_{k+1}| \leq 2\alpha\epsilon. \tag{13}$$

Lemma 4 requires $|\sum_{i=1}^k (x_i - y_i)|$ to be uniformly bounded so that the difference $|\lambda_k - \mu_k|$ is also bounded. Hence, it is interesting to know under which conditions it is possible to construct a sequence $\{y_k\} \in Y \subseteq \mathbb{R}^n$ that stays "close" to an arbitrary sequence $\{x_k\} \in X \subseteq \mathbb{R}^n$. Next we show that this is always possible when Y is a finite collection of points from $\mathbb{R}^n$ and $X \subseteq \text{conv}(Y)$, the convex hull of Y.

A. Construction of Discrete Sequences

Consider a set of m discrete points Y from $\mathbb{R}^n$ such that $X \subseteq \text{conv}(Y)$. Collect the points in Y as columns in matrix W. We can write any point $x \in X$ as a convex combination of points in Y, *i.e.*, $x = Wu$ where $u \in U := \{u \in \mathbb{R}^m \mid \|u\|_1 = 1, u \succeq 0\}$. Note there might exist multiple vectors in U such that $Wu = x$[1]. Similarly, define $E := \{e \in \{0,1\}^m \mid \|e\|_1 = 1\}$ to be the set containing the m-dimensional standard basis vectors. There is only one vector $e \in E$ for each $y \in Y$ such that $We = y$. We can write $\sum_{i=1}^k (x_i - y_i) = W \sum_{i=1}^k (u_i - e_i)$, and since $|W \sum_{i=1}^k (u_i - e_i)| \leq \|W\|_2 \| \sum_{i=1}^k (u_i - e_i)\|_2$ by the Cauchy-Schwarz inequality it follows that showing that

$$\Big\| \sum_{i=1}^k (u_i - e_i) \Big\|_2 \tag{14}$$

is bounded for all k is sufficient to establish the boundedness of $\sum_{i=1}^k (x_i - y_i)$. We show this in the following corollary, which is a straightforward extension of [13, Theorem 4].

Corollary 2 (Block Sequence)**.** *Let* $T \in \mathbb{N}$*,* $U := \{u \in \mathbb{R}^m \mid \|u\|_1 = 1, u \succeq 0\}$*,* $E := \{e \in \{0,1\}^m \mid \|e\|_1 = 1\}$ *and define* $S = \sum_{i=1}^{Tm} u_i$ *where* $u_i \in U$ *for all* $i = 1, \dots, Tm$*. Let* $\delta \in D := \{w \in \mathbb{R}^m \mid w^T \mathbf{1} = 0, \|w\|_\infty \leq 1\}$ *and select*

$$e_k \in \arg\min_{e \in E} \Big\| \delta + S - \sum_{i=1}^{k-1} e_i - e \Big\|_\infty. \tag{15}$$

Then, we have that $(\delta + S - S') \in D$ *where* $S' = \sum_{i=1}^{Tm} e_i$*.*

[1] One such vector can be efficiently found by solving convex optimisation problem $\min_{u \in U} \ \|Wu - x\|$ with standard convex methods – see Chapters 6 and 9 in [16].

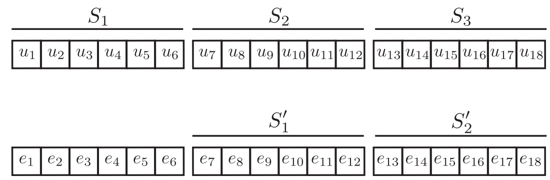

Fig. 4. Illustrating the construction of a subsequence in an online mode in a "delayed" or "shifted" manner with $m = 3$ and $T = 2$. Subsequence $\{e_7, \dots, e_{12}\}$ is constructed with update (15) with subsequence $\{u_1, \dots, u_6\}$, and therefore $(\delta + \sum_{i=1}^6 u_i - \sum_{j=7}^{12} e_j) \in D$. Subsequence $\{e_1, \dots, e_6\}$ can be selected randomly from E and it does not affect the boundedness of (14).

Corollary 2 says that for any subsequence $\{u_1, \dots, u_{Tm}\}$ we can use update (15) to construct a subsequence $\{e_1, \dots, e_{Tm}\}$ such that

$$(\delta + S - S') = \left(\delta + \sum_{i=1}^{Tm} (u_i - e_i) \right) := \tilde{\delta} \in D. \tag{16}$$

Hence, for any other subsequence $\{u_{Tm+1}, \dots, u_{2Tm}\}$ we can use update (15) to construct a subsequence $\{e_{Tm+1}, \dots, e_{2Tm}\}$ such that

$$\left(\delta + \sum_{i=1}^{2Tm} (u_i - e_i) \right) = \left(\tilde{\delta} + \sum_{i=Tm+1}^{2Tm} (u_i - e_i) \right) \in D.$$

Next, observe that if we let $S_l = \sum_{j=(l-1)Tm+1}^{l(Tm)} u_j$ and $S'_l = \sum_{j=(l-1)Tm+1}^{l(Tm)} e_j$ with $l = 1, 2, \dots$ we will always have that $\{\sum_{i=1}^{l} (S_i - S'_i)\}$ is a sequence of points from $D := \{w \in \mathbb{R}^m \mid w^T \mathbf{1} = 0, \|w\|_\infty \leq 1\}$ and therefore bounded. The boundedness of (14) for all $k = 1, 2, \dots$ follows immediately from the fact that subsequences have finite length.

Importantly, note that update (15) requires to know Tm elements from U in *advance* in order to select Tm elements from E. This implies that it will only be possible to construct subsequences in an online mode in a "delayed" or "shifted" manner – see the example in Figure 4. Nonetheless, since subsequences have finite length, constructing subsequences in a "shifted" manner does not affect the boundedness of (14).

An important observation is that the elements in a subsequence $\{e_1, \dots, e_{Tm}\}$ can be reordered and still have that $(\delta + \sum_{i=1}^{Tm} (u_i - e_i)) \in D$. This will be particularly useful in problems where there are constraints on the order how actions y_k (and so e_k) can be selected. In the next section we present a network example where the selection of discrete actions has constraints and cannot be solved with classic max-weight approaches.

IV. Wireless Network Example

A. Problem Setup

Consider the network shown in Figure 5, an Access Point (AP) that transmits to two wireless nodes. Time is slotted and in each time slot packets arrive at the queues of the AP (Q_1 and Q_2) with probability p_1 and p_2, respectively. In each time slot the AP takes an action from action set $Y := \{y_0, y_1, y_2\} = \{[0,0], [1,0], [0,1]\}$, where each action corresponds, respectively, to not transmitting, to transmitting

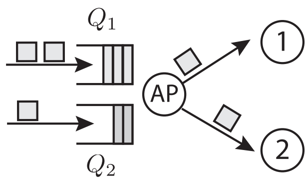

Fig. 5. Illustrating the network in the example of Section IV. The Access Point (AP) sends packets from Q_1 to node 1, and from Q_2 to node 2.

one packet from Q_1 to node 1, and to transmitting one packet from Q_2 to node 2.

The transmission protocol of the AP has constraints on how actions can be selected. In particular, it is not possible to select action y_1 after y_2 without first selecting y_0. Likewise, it is not possible to select y_2 after y_1 without first selecting y_0. However, y_1 or y_2 can be selected sequentially. An example of an admissible sequence is $\{y_1, y_1, y_1, y_0, y_2, y_0, y_1, y_1, y_0, y_2, y_2, \dots\}$. These type of constraints appear in different areas, and are known in the literature as reconfiguration delays [7]. For example, asking for the Channel State Information (CSI) in wireless communications in order to adjust the transmission parameters[2].

Our goal is to design a scheduling policy for the AP (select actions from set Y) in order to minimise a convex cost function of the average throughput $\bar{x}_k$, and ensure that the system is stable, *i.e.*, the queues do not overflow and so all traffic can be served.

B. Problem Formulation

The convex or fluid formulation of the problem is

$$\underset{x \in X}{\text{minimise}} \quad f(x) \tag{17}$$

$$\text{subject to} \quad b \preceq x \tag{18}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$, $b \in \mathbb{R}^2_+$ and X is a bounded convex subset from $\operatorname{conv}(Y) \subset \mathbb{R}^2$ that depends on the protocol constraints, *i.e.*, on how actions can be selected. If there were no constraints on the order in which actions could be selected we would have $X := \operatorname{conv}(Y)$, however, this is *not* the case. Characterising X is as simple as noting that if we have a subsequence of actions where y_1 and y_2 appear (each) sequentially, then y_0 should appear *at least* twice in order to have a subsequence that is compliant with the transmission protocol – for example $\{y_0, y_1, y_1, y_1, y_0, y_2, y_2, y_2, y_2\}$. Conversely, any subsequence in which y_0 appears at least twice can be reordered to obtain a subsequence that is compliant with the transmission protocol. Since we can always choose a subsequence of discrete actions and then reorder its elements (see discussion after Corollary 2), we just need to enforce that y_0 appears twice in a subsequence. This can be obtained if the convex combination of a point $x \in X$ uses point y_0 at least fraction $2/Tm$, where (Tm) is the length of a subsequence. This will be the case when

[2]The CSI in wireless communications is in practice requested periodically, and not only at the beginning of a transmission, but we will assume this for simplicity of exposure. The extension is nevertheless straightforward.

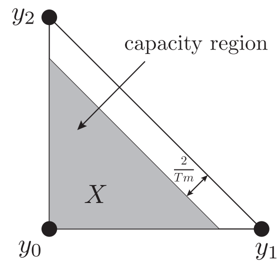

Fig. 6. Network capacity region of the example in Section IV. $X = \eta \operatorname{conv}(Y)$ with $\eta = 1 - 2/(Tm)$.

$$X := \eta \operatorname{conv}(Y), \tag{19}$$

where $\eta := 1 - 2/(Tm)$. Note from (19) that as T gets larger $\eta \to 1$ and therefore $X \to \operatorname{conv}(Y)$, *i.e.*, the network capacity region increases. Figure 6 illustrates the network capacity region of the example.

1) Updates: We use scaled queues occupancies as approximate multipliers. At each time slot we have updates

$$x_k \in \arg\min_{x \in X} \{f(x) - \alpha Q^T x\}, \tag{20}$$

$$Q_{k+1} = [Q_k - y_k + B_k]^+, \tag{21}$$

where B_k are i.i.d. random variables that take values $\{0, 1\}$ with mean $[p_1, p_2]^T$. Action $y_k = We_k$, and e_k is obtained with (15) in the online mode explained in Section III-A. The elements in a subsequences are reordered to comply with the transmission protocol constraints.

2) Simulation: We run a simulation with objective function $f = \|x\|_2^2$ and parameters $p_1 = 0.25$, $p_2 = 0.5$, $\alpha = 0.05$, $\lambda_1 = \alpha Q_1 = 0$, $T = 3$ (so the number of elements in a subsequence is 9 since $m = 3$). Figure 7 shows the convergence of $\bar{\lambda}_k$ to a ball around $\lambda^\star$. Interestingly, see that whereas $\bar{\lambda}_k$ converges to $\lambda^\star$, the Lagrange multipliers λ_k (grey points) do not converge to a ball around $\lambda^\star$. The stability of the system follows from the fact that if the average of the Lagrange multipliers is bounded, then the average of the α-scaled queue occupancies is also bounded. Figure 8 shows that $f(\bar{x}_k)$ converges to a ball around $f^\star$ as k increases.

V. Conclusions

We have shown that the stochastic subgradient method for the dual problem in convex optimisation allows the use of approximate Lagrange multipliers. The use of approximate Lagrange multipliers is motivated because it is possible to identify α-scaled queue occupancies as approximations of Lagrange multipliers in the subgradient method. As a result, it is possible to cast some discrete decision making problems as convex optimisations. One of the advantages of our convex approach is that we can make optimal control decisions that have ordering constraints.

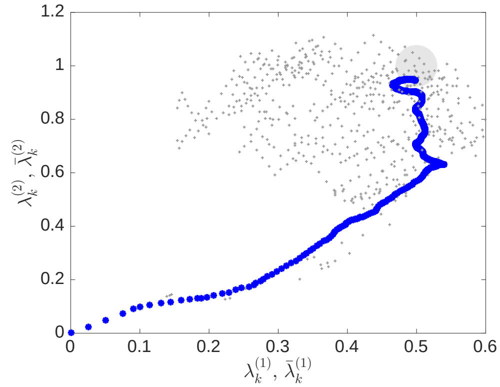

Fig. 7. Illustrating the convergence of average $\bar{\lambda}_{k+1}$ (blue line) to a ball around $\lambda^\star = [0.5, 1]^T$ (grey area). Grey points indicate the values of the Lagrange multipliers λ_k, and $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ denote the first and second components of vector λ_k.

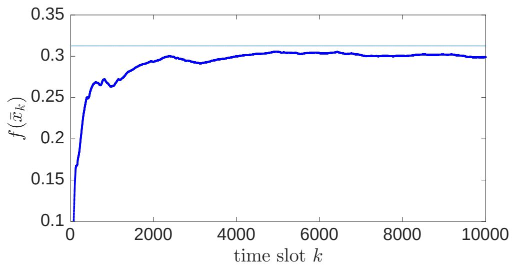

Fig. 8. Illustrating the convergence of $f(\bar{x}_k)$ to $f^\star = 0.3125$ (straight line) in the network example of Section IV.

Appendix

A. Proof of Lemma 1

Consider update $\lambda_{k+1} = [\lambda_k + \alpha g(x_k)]^+$ and suppose we select $x_k \in X(\mu_k) := \{x \in X \mid L(x, \mu_k) - q(\mu_k) \leq \epsilon\}$. Observe that

$$\begin{aligned}
&\|\lambda_{k+1} - \lambda^\star\|_2^2 \\
&= \|[\lambda_k + \alpha g(x_k)]^+ - \lambda^\star\|_2^2 \\
&\leq \|\lambda_k + \alpha g(x_k) - \lambda^\star\|_2^2 \\
&= \|\lambda_k - \lambda^\star\|_2^2 + \alpha^2 \|g(x_k)\|_2^2 + 2\alpha(\lambda_k - \lambda^\star)^T g(x_k) \\
&\leq \|\lambda_k - \lambda^\star\|_2^2 + \alpha^2 L_g^2 + 2\alpha(q(\lambda_k) + \epsilon - q(\lambda^\star)),
\end{aligned}$$

where the last inequality follows from the fact that $(\lambda_k - \lambda^\star)^T g(x_k) = L(x_k, \lambda_k) - L(x_k, \lambda^\star) \leq L(x_k, \lambda_k) - q(\lambda^\star) \leq q(\lambda_k) + \epsilon - q(\lambda^\star)$, and $\|g(x_k)\|_2 \leq L_g$ for all k. Rearranging terms yields

$$\begin{aligned}
&\|\lambda_{k+1} - \lambda^\star\|_2^2 - \|\lambda_k - \lambda^\star\|_2^2 && (22)\\
&\quad \leq \alpha^2 L_g^2 + 2\alpha(q(\lambda_k) + \epsilon - q(\lambda^\star)). && (23)
\end{aligned}$$

Now let $\Lambda_\delta := \{\lambda \in \mathbb{R}_+^m \mid q(\lambda^\star) - q(\lambda) \leq \alpha L_g^2 + \epsilon\}$ and consider two cases. Case (i) ($\lambda_k \notin \Lambda_\delta$) Observe that (23) is strictly negative and therefore λ_k is monotonically attracted to Λ_δ. Hence, for any $\lambda_1 \in \mathbb{R}^m$ and k sufficiently large it will eventually happen that $\lambda_k \in \Lambda_\delta$ and therefore $|q(\lambda_k) - q(\lambda^\star)| \leq \alpha L_g^2/2 + \epsilon$. Case (ii) ($\lambda_k \in \Lambda_\delta$) We cannot ensure that the RHS in (23) is strictly negative, however, from the Lipchitz continuity of the dual function we have $|q(\lambda_{k+1}) - q(\lambda_k)| \leq \|\lambda_{k+1} - \lambda_k\|_2 L_g = \|[\lambda_k + \alpha g(x_k)]^+ - \lambda_k\|_2 L_g \leq \|\alpha g(x_k)\|_2 L_g \leq \alpha L_g^2$. Now see that since $|q(\lambda_{k+1}) - q(\lambda^\star)| = |q(\lambda_{k+1}) - q(\lambda_k) + q(\lambda_k) - q(\lambda^\star)| \leq |q(\lambda_{k+1}) - q(\lambda_k)| + |q(\lambda_k) - q(\lambda^\star)| \leq \alpha L_g^2 + \alpha L_g^2/2 + \epsilon = \alpha(3/2)L_g^2 + \epsilon$. That is, $\lambda_{k+1} \in \Lambda'_\delta := \{\lambda \in \mathbb{R}_+^m \mid q(\lambda^\star) - q(\lambda) \leq \alpha(3/2)L_g^2 + \epsilon\}$. Finally, note that $\Lambda_\delta \subset \Lambda'_\delta$ and that since from case (i) any $\lambda_{k+1} \in \Lambda'_\delta \setminus \Lambda_\delta$ will be attracted to Λ_δ, we can conclude that for every $\lambda_1 \in \mathbb{R}^m$ there exists a non-negative and non-increasing sequence $\{\beta_k\}$ that converges to $\alpha(3/2)L_g^2 + \epsilon$ for k sufficiently large and $-\beta_k \leq q(\lambda_k) - q(\lambda^\star) \leq 0$ for all k.

B. Proof of Lemma 2

Observe that $\lambda_{k+1} = [\lambda_k + \alpha g(x_k)]^+ \succeq \lambda_k + \alpha g(x_i)$, and rearranging terms $\lambda_{k+1} - \lambda_k \succeq \alpha g(x_k)$. Summing from $i = 1, \dots, k$ yields $\alpha \sum_{i=1}^k g(x_i) \preceq \lambda_{k+1} - \lambda_1 \preceq \lambda_{k+1}$. Dividing by αk and using the fact that $g(\bar{x}_k) \preceq \frac{1}{k}\sum_{i=1}^k g(x_i)$ yields the stated result.

C. Proof of Theorem 1

From Lemma 1 we have that $-\beta_k \leq q(\lambda_k) - q(\lambda^\star) \leq 0$, where $\{\beta_k\}$ is a non-increasing sequence that converges to $\alpha(3/2)L_g^2 + \epsilon$ for k sufficiently large. Next, since $q(\lambda_k) - q(\lambda^\star) \leq L(\bar{x}_k, \lambda_k) - q(\lambda^\star) = f(\bar{x}_k) + \lambda_k^T g(\bar{x}_k) - q(\lambda^\star)$, by rearranging terms and using the fact that $q(\lambda^\star) = f^\star$ (by strong duality) we obtain $-\beta_k - \lambda_k^T g(\bar{x}_k) \leq f(\bar{x}_k) - f^\star$. From Lemma 2 we have that $g(\bar{x}_k) \preceq \lambda_{k+1}/(\alpha k)$ for all k, and since $\|\lambda_{k+1}\|_2 \leq \|\lambda_k\|_2 + \alpha L_g$ we obtain

$$-\beta_k - \frac{\lambda_k^\circ}{\alpha k} \leq f(\bar{x}_k) - f^\star \tag{24}$$

where $\lambda_k^\circ = \lambda_k^T \lambda_{k+1}$.

We now show the upper bound in (7). Since $f^\star = q(\lambda^\star) \geq \frac{1}{k}\sum_{i=1}^k q(\lambda_i) = \frac{1}{k}\sum_{i=1}^k f(x_i) + \lambda_i^T g(x_i) \geq f(\bar{x}_k) + \frac{1}{k}\sum_{i=1}^k \lambda_i^T g(x_i)$, by rearranging terms we have that $f(\bar{x}_k) - f^\star \leq -\frac{1}{k}\sum_{i=1}^k \lambda_i^T g(x_i)$. Now observe that for any sequence $\{x_k\}$ in X we can write $\|\lambda_{k+1}\|_2^2 = \|[\lambda_k + \alpha g(x_k)]^+\|_2^2 \leq \|\lambda_k + \alpha g(x_k)\|_2^2 \leq \|\lambda_k\|_2^2 + \alpha^2\|g(x_k)\|_2^2 + 2\alpha\lambda_k^T g(x_k)$. Applying the latter equation recursively for $i = 1, \dots, k$ we obtain that $\|\lambda_{k+1}\|_2^2 \leq \|\lambda_1\|_2^2 + \alpha^2 \sum_{i=1}^k \|g(x_i)\|_2^2 + 2\alpha \sum_{i=1}^k \lambda_i^T g(x_i)$, and rearranging terms and dividing by $2\alpha k$ yields $-\frac{1}{k}\sum_{i=1}^k \lambda_i^T g(x_i) \leq \frac{\alpha}{2k}\sum_{i=1}^k \|g(x_i)\|_2^2 + \frac{\|\lambda_1\|_2^2}{2\alpha k}$. Finally, since $\|g(x_i)\|_2 \leq L_g$ by Assumption 1 we obtain

$$f(\bar{x}_k) - f^\star \leq \frac{\alpha L_g^2}{2} + \frac{\|\lambda_1\|_2^2}{2\alpha k},$$

which concludes the proof.

D. Proof of Lemma 3

In each iteration we aim to decrease the *expected* distance of λ_k from $\lambda^\star$. In short, observe that for any vector $\theta \in \mathbb{R}^m$ we have $\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2 \mid \lambda_k) = \mathbb{E}(\|[\lambda_k + \alpha\tilde{g}(x_k)]^+ - \theta\|_2^2 \mid \lambda_k) \leq \mathbb{E}(\|\lambda_k + \alpha\tilde{g}(x_k) - \theta\|_2^2 \mid \lambda_k)$ where $\tilde{g}(x_k) = g(x_k) + B_k$.

Expanding the RHS of the last equation and rearranging terms yields

$$\begin{aligned}\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2 \mid \lambda_k) - \|\lambda_k - \theta\|_2^2 \quad (25)\\ \leq \alpha^2 \mathbb{E}(\|\tilde{g}(x_k)\|_2^2 \mid \lambda_k) + 2\alpha \mathbb{E}((\lambda_k - \theta)^T \tilde{g}(x_k) \mid \lambda_k).\\ \leq \alpha^2 L_h^2 + 2\alpha \mathbb{E}((\lambda_k - \theta)^T \tilde{g}(x_k) \mid \lambda_k). \quad (26)\end{aligned}$$

where the last equation follows from the boundedness of the subgradient. Next, see that since $\mathbb{E}((\lambda_k - \theta)^T (g(x_k) + B_k) \mid \lambda_k) = (\lambda_k - \theta)^T (g(x_k) + b) = L(x_k, \lambda_k) - L(x_k, \theta) \leq L(x_k, \lambda_k) - q(\theta) \leq q(\lambda_k) + \epsilon - q(\theta)$ we have that $\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2 \mid \lambda_k) - \|\lambda_k - \theta\|_2^2 \leq \alpha^2 L_h^2 + 2\alpha(q(\lambda_k) + \epsilon - q(\theta))$ and taking expectations yields

$$\begin{aligned}\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2) - \mathbb{E}(\|\lambda_k - \theta\|_2^2)\\ \leq \alpha^2 L_h^2 + 2\alpha \mathbb{E}(q(\lambda_k) + \epsilon - q(\theta)).\end{aligned}$$

Applying the latter expression recursively to obtain $\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2) - \|\lambda_1 - \theta\|_2^2 \leq \alpha^2 k L_h^2 + 2\alpha \sum_{i=1}^k \mathbb{E}(q(\lambda_i) + \epsilon - q(\theta))$. Dropping $\mathbb{E}(\|\lambda_{k+1} - \theta\|_2^2)$, further rearranging and dividing by $2\alpha k$ yields

$$-\frac{\|\lambda_1 - \theta\|_2^2}{2\alpha k} - \frac{\alpha L_h^2}{2} - \epsilon \leq \frac{1}{k} \sum_{i=1}^k \mathbb{E}(q(\lambda_i) - q(\theta)). \quad (27)$$

Next, see that by the linearity of the expectation one can write

$$\frac{1}{k} \sum_{i=1}^k \mathbb{E}(q(\lambda_i)) = \mathbb{E}\left(\frac{1}{k} \sum_{i=1}^k q(\lambda_i)\right)$$

where $\lambda_i,\ i = 1, \ldots, k$ are the Lagrange multiplier obtained from update (9). Next, let $\bar{\lambda}_k := \frac{1}{k} \sum_{i=1}^k \lambda_i$ and see that by the concavity of q we can write $\mathbb{E}(\frac{1}{k} \sum_{i=1}^k q(\lambda_i)) \leq \mathbb{E}(q(\bar{\lambda}_k)) \leq q(\mathbb{E}(\bar{\lambda}_k))$. Hence,

$$-\frac{\|\lambda_1 - \theta\|_2^2}{2\alpha k} - \frac{\alpha L_h^2}{2} - \epsilon \leq q(\mathbb{E}(\bar{\lambda}_k)) - q(\theta),$$

and if we let $\theta = \lambda^\star$ we obtain that $q(\mathbb{E}(\bar{\lambda}_k))$ converges monotonically to a ball around $q(\lambda^\star)$.

E. Proof of Theorem 2

We follow the same steps as in the proof of Theorem 1. From the stochastic subgradient method (see Lemma 3) we have that for any $\theta \in \mathbb{R}^m$ there exists a non-negative and strictly decreasing sequence $\{\beta_k\}$ such that $-\beta_k \leq q(\mathbb{E}(\bar{\lambda}_k)) - q(\theta)$ for all k. Also, since we can always write $q(\mathbb{E}(\bar{\lambda}_k)) \leq L(\mathbb{E}(\bar{x}_k), \mathbb{E}(\bar{\lambda}_k)) = f(\mathbb{E}(\bar{x}_k)) + \mathbb{E}(\bar{\lambda}_k)^T g(\mathbb{E}(\bar{x}_k))$, and $g(\mathbb{E}(\bar{x}_k)) \preceq \mathbb{E}(\lambda_{k+1})/(\alpha k)$ from Lemma 2, rearranging terms yields

$$-\beta_k - \frac{\mathbb{E}(\lambda_k^\circ)}{\alpha k} \leq f(\mathbb{E}(\bar{x}_k)) - q(\theta). \quad (28)$$

where $\mathbb{E}(\lambda_k^\circ) = \mathbb{E}(\bar{\lambda}_k)^T \mathbb{E}(\lambda_{k+1})$.

We now proceed to upper bound (28). From (27) we have $\frac{1}{k} \sum_{i=1}^k \mathbb{E}(q(\lambda_i)) - q(\theta) = \frac{1}{k} \sum_{i=1}^k \mathbb{E}(f(x_i) + \lambda_i^T (g(x_i) + b)) - q(\theta) \geq f(\mathbb{E}(\bar{x}_k)) + \mathbb{E}(\frac{1}{k} \sum_{i=1}^k \lambda_i^T (g(x_i) + b)) - q(\theta)$ and therefore

$$f(\mathbb{E}(\bar{x}_k)) - q(\lambda^\star(b)) \leq -\mathbb{E}\left(\frac{1}{k} \sum_{i=1}^k \lambda_i^T (g(x_i) + b)\right) \quad (29)$$

where $\lambda^\star(b)$ depends on b. Now observe that using $\theta = 0$ in (26) we have

$$\mathbb{E}(\|\lambda_{k+1}\|_2^2 \mid \lambda_k) - \|\lambda_k\|_2^2 \leq \alpha^2 L_h^2 + 2\alpha \mathbb{E}(\lambda_k^T (g(x_k) + b)).$$

Applying the latter equation recursively we obtain $\mathbb{E}(\|\lambda_{k+1}\|_2^2) \leq \|\lambda_1\|_2^2 + \alpha^2 k L_h^2 + 2\alpha \mathbb{E}(\sum_{i=1}^k \lambda_i^T (g(x_i) + b))$. Rearranging terms and dividing by $2\alpha k$ yields

$$-\mathbb{E}\left(\frac{1}{k} \sum_{i=1}^k \lambda_i^T (g(x_i) + b)\right) \leq \frac{\alpha L_h^2}{2} + \frac{\|\lambda_1\|_2^2}{2\alpha k}. \quad (30)$$

Finally, letting $\theta = \lambda^\star(b)$ in (28) concludes the proof.

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