ON THE IDENTIFICATION OF NONLINEAR SYSTEMS BY COMBINING IDENTIFIED LINEAR MODELS

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Abstract
Divide and conquer identification approaches are considered with the aim of permitting well-developed linear methods can be brought to bear on the nonlinear identification task. Transfer functions of the plant linearisations are identified from measured data and the requirement is then to infer the underlying nonlinear system. It is shown that knowledge only of the transfer functions of the linearisations of the nonlinear system is insufficient to permit such reconstruction and sufficient conditions based on augmented transfer function knowledge are derived.

1. Introduction

Mathematical models of dynamic systems are required in a wide range of applications. These models may be determined directly from measured experimental data (when, for example, the expense of developing a detailed analytic model cannot be justified), derived analytically from first principles or, perhaps most commonly, determined by some combination of empirical and analytic methods. It should be noted that even in the case of models derived purely by analytic methods, experimental external validation is required in order to establish their accuracy and range of applicability. The identification of linear systems from measured experimental data has received considerable attention over the last thirty years and there exists a wealth of theoretical results relating to issues such as structure identification, parameter estimation, experiment design and model validation testing together with a great deal of accumulated practical experience. However, all systems are in reality nonlinear and identification techniques are less well developed for systems which cannot be accurately approximated by a single linear time-invariant system. This type of situation exists not only in the identification field but also more generally. Whilst nonlinear dynamic systems are widespread, the analysis and design of such systems remains relatively difficult. In contrast, although systems with genuinely linear time-invariant dynamics do not, in reality, exist, techniques for the analysis and design of linear time-invariant systems are rather better developed. It is, therefore, often attractive to consider a divide and conquer strategy whereby the analysis/design of a nonlinear system is decomposed into the analysis/design of a collection of linear time-invariant systems. In the context of control system analysis and design, this type of strategy is well established and forms the basis, for example, of one of the most widely, and successfully, applied techniques for the design of nonlinear controllers; namely, gain-scheduling. Similarly, in the context of system identification it is common practice, when faced with the task of modelling a nonlinear system, to initially identify a number of linear approximations to the system each of which is locally valid.

Traditionally, the first-order Taylor series expansion of a nonlinear system is often employed as a local linear approximation. However, since the first-order expansion is linear only when the expansion is carried out relative to an equilibrium point, consideration is necessarily confined to near equilibrium operation. Of course, while the dynamics in the vicinity of a single equilibrium point are clearly important, the dynamic behaviour during rapid transitions between equilibrium points and, indeed, the behaviour during sustained operation far from equilibrium are also frequently of considerable interest. Fortunately, this issue is addressed by a recent generalisation of the conventional equilibrium linearisation, namely the velocity-based linearisation (Leith & Leithead 1998a,b). In contrast to the conventional series expansion linearisation approach, the velocity-based approach associates a linear system, namely the velocity-based linearisation, with *every* operating point of a nonlinear system (including those far from equilibrium) not just the equilibrium operating points. The solution to the velocity-based linearisation associated with an operating point locally approximates the solution of the nonlinear system and the global solution to the nonlinear system can be recovered by appropriately piecing together the solutions to its velocity-based linearisations. While maintaining continuity with linear methods, the velocity-based approach removes the restriction to near equilibrium operation which is inherent to conventional linearisation approaches and accommodates, for example, both transitions between equilibrium operating points and sustained operation far from equilibrium. Since it describes the dynamics at every operating point, the velocity-based linearisation family associated with a nonlinear system is alternative representation of the system and involves no loss of information.

Whilst originally derived in the context of control system analysis and design, the velocity-based linearisation provides a natural framework within which to consider divide and conquer identification approaches whereby well-developed linear methods can be brought to bear on the nonlinear identification task. In the context of system identification, the transfer functions of the velocity-based linearisations might be identified from measured data using well-established linear methods and the requirement is then to infer the velocity-based linearisation family or, equivalently, the nonlinear system. It is the latter task which is the subject of the present paper. .

2. Velocity-based linearisation

The velocity-based analysis and design representation is briefly summarised. Consider a nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{1}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable nonlinear functions and $\mathbf{r} \in \Re^m$ denotes the input to the plant, $\mathbf{y} \in \Re^p$ the output and $\mathbf{x} \in \Re^n$ the states. Differentiating (1), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}}, \quad \dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{2}$$

The velocity-based formulation, (2), is dynamically equivalent to (1) in the sense that, for appropriate initial conditions, they have the same solution, $\mathbf{x}$. It can be shown (Leith & Leithead 1998a) that the solution $\hat{\mathbf{x}}$ to the linear system (the "velocity-based linearisation")

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}}, \ \dot{\hat{\mathbf{w}}} = \nabla_x\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}} + \nabla_r\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}}, \ \dot{\hat{\mathbf{y}}} = \nabla_x\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}} + \nabla_r\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \tag{3}$$

approximates the solution $\mathbf{x}$ to the nonlinear system locally to the operating point $(\mathbf{x}_1, \mathbf{r}_1)$. Since a linear system (3) is associated with every operating point of the nonlinear system, there is a family of velocity-based linearisations associated with the nonlinear system. Whilst the solution to a single velocity-based linearisation is only a local approximation to the solution of the nonlinear system, the solutions to the members of this family can be pieced together to recover the solution of the nonlinear system. . The direct relationship between the linearisation, (3), and the velocity-based nonlinear system, (2), is clear; namely, the velocity-based linearisation is obtained by simply "freezing" (2) at the relevant operating point.

3. Conventional transfer function knowledge alone is insufficient

Inferring the velocity-based linearisation family from the corresponding family of transfer functions is not quite as straightforward as might at first appear. Some indication of this might be evident from the observation that, although it is common practice to identify the equilibrium linearisations of a nonlinear system, rarely is there any attempt to then combine these linearisations in a rigorous manner in order to recover a description of the nonlinear dynamics as the system moves from the vicinity of one equilibrium point to the vicinity of another. The crux of the problem is that only input-output data is available and so there exist infinitely many choices of state-space realisation of each identified transfer function. When piecing together the solutions to the linear systems in order to recover the solutions to the nonlinear system, this piecing together is carried out in state-space. It is therefore necessary to determine the appropriate choice of state for each linear system which ensures compatibility with the underlying nonlinear system.

That knowledge only of the transfer functions of the velocity-based linearisations is insufficient to enable the underlying nonlinear system to be recovered uniquely can be seen from the following analysis. The nonlinear system (1) has, at the operating point $(\mathbf{x}_1, \mathbf{r}_1)$, the velocity-based linearisation **Error! Reference source not found.**-(3). Of course, the dynamics of a linear system are invariant under a non-singular state-transformation. Consider, therefore, the nonlinear system for which the velocity-based linearisation, at the operating point $(\mathbf{x}_1, \mathbf{r}_1)$, is

$$\dot{\hat{\boldsymbol{\chi}}} = \mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\boldsymbol{\omega}}, \quad \dot{\hat{\boldsymbol{\omega}}} = \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\nabla_x\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\boldsymbol{\omega}} + \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\nabla_r\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \tag{4}$$

$$\dot{\hat{\mathbf{v}}} = \nabla_x\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\mathbf{T}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\boldsymbol{\omega}} + \nabla_r\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \tag{5}$$

where $\boldsymbol{\chi}, \boldsymbol{\omega}, \hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\omega}} \in \Re^n$ and $\mathbf{T}(\bullet, \bullet)$ is a uniformly bounded non-singular matrix which is differentiable with uniformly bounded derivatives. It can be seen that the velocity-based linearisations, **Error! Reference source not found.**-(3) and (15)-(13), are related by the non-singular transformation $\hat{\boldsymbol{\omega}} = \mathbf{T}^{-1}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}}, \ \hat{\boldsymbol{\chi}} = \hat{\mathbf{x}}, \ \hat{\mathbf{v}} = \hat{\mathbf{y}}$ (6)

and so are dynamically equivalent. Similarly for the velocity-based linearisations at other operating points. The velocity-form of the nonlinear system with linearisation family (15)-(13) is

$$\dot{\boldsymbol{\chi}} = \mathbf{T}(\boldsymbol{\chi}, \mathbf{r})\boldsymbol{\omega}, \quad \dot{\boldsymbol{\omega}} = \mathbf{T}^{-1}(\boldsymbol{\chi}, \mathbf{r})\nabla_x\mathbf{F}(\boldsymbol{\chi}, \mathbf{r})\mathbf{T}(\boldsymbol{\chi}, \mathbf{r})\boldsymbol{\omega} + \mathbf{T}^{-1}(\boldsymbol{\chi}, \mathbf{r})\nabla_r\mathbf{F}(\boldsymbol{\chi}, \mathbf{r})\,\dot{\mathbf{r}} \tag{7}$$

$$\dot{\mathbf{v}} = \nabla_x\mathbf{G}(\boldsymbol{\chi}, \mathbf{r})\mathbf{T}(\boldsymbol{\chi}, \mathbf{r})\boldsymbol{\omega} + \nabla_r\mathbf{G}(\boldsymbol{\chi}, \mathbf{r})\,\dot{\mathbf{r}} \tag{8}$$

Letting $\boldsymbol{\omega} = \mathbf{T}^{-1}(\boldsymbol{\chi}, \mathbf{r})\mathbf{z}$ the nonlinear system, (7)-(8), may be reformulated as

$$\dot{\boldsymbol{\chi}} = \mathbf{z}, \quad \dot{\mathbf{z}} = \nabla_{\mathbf{x}}\mathbf{F}(\boldsymbol{\chi}, \mathbf{r})\mathbf{z} + \nabla_{\mathbf{r}}\mathbf{F}(\boldsymbol{\chi}, \mathbf{r})\,\dot{\mathbf{r}} + \boldsymbol{\varepsilon} \tag{9}$$

$$\dot{\boldsymbol{\nu}} = \nabla_{\mathbf{x}}\mathbf{G}(\boldsymbol{\chi}, \mathbf{r})\mathbf{z} + \nabla_{\mathbf{r}}\mathbf{G}(\boldsymbol{\chi}, \mathbf{r})\,\dot{\mathbf{r}} \tag{10}$$

where $\boldsymbol{\varepsilon} = \dot{\mathbf{T}}(\boldsymbol{\chi}, \mathbf{r})\,\mathbf{T}^{-1}(\boldsymbol{\chi}, \mathbf{r})\mathbf{z}$. Despite the dynamic equivalence of the members of the velocity-based linearisation families, it is evident that the dynamics of the nonlinear systems, **Error! Reference source not found.**-(2) (equivalently (1)) and (7)-(8), are *not* the same. The difference between the dynamics is embodied by the perturbation term, $\boldsymbol{\varepsilon}$, and arises from the variation of the state transformation, (19), with the operating point.

4. Conditions for Reconstructing a Nonlinear System from its Identified Linearisations

It is evident from the foregoing analysis that additional information is required in order to permit the nonlinear system associated with a family of identified transfer functions to be reconstructed. Various types of information might, of course, provide the required additional information but consideration here is confined to a simple, but effective, extension of the available transfer function information.

Before proceeding, it is useful to reformulate the nonlinear system, (1), as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\boldsymbol{\rho}), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\boldsymbol{\rho}) \tag{11}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are nonlinear functions and $\boldsymbol{\rho}(\mathbf{x},\mathbf{r}) \in \Re^q$, $q \le m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ constant. Trivially, this reformulation can always be achieved by letting $\boldsymbol{\rho} = [\mathbf{x}^T \ \ \mathbf{r}^T]^T$, in which case q=m+n. However, the nonlinearity of the system is frequently dependent on only a subset of the states and inputs, in which case the dimension, q, of $\boldsymbol{\rho}$ is less than m+n.

Consider two nonlinear systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\boldsymbol{\rho}), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\boldsymbol{\rho}) \tag{12}$$

and

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{r} + \tilde{\mathbf{f}}(\tilde{\boldsymbol{\rho}}), \quad \tilde{\mathbf{y}} = \tilde{\mathbf{C}}\tilde{\mathbf{x}} + \tilde{\mathbf{D}}\mathbf{r} + \tilde{\mathbf{g}}(\tilde{\boldsymbol{\rho}}) \tag{13}$$

where $\dot{\mathbf{r}} \in \Re^m$, $\mathbf{w}, \tilde{\mathbf{w}} \in \Re^n$, $\mathbf{y}, \tilde{\mathbf{y}} \in \Re^p$, $\boldsymbol{\rho}$, $\tilde{\boldsymbol{\rho}} \in \Re^q$. Differentiate to obtain the corresponding velocity-based nonlinear systems

$$\begin{aligned} &\dot{\boldsymbol{\rho}} = \nabla_{\mathbf{x}}\boldsymbol{\rho}\,\mathbf{w} + \nabla_{\mathbf{r}}\boldsymbol{\rho}\,\dot{\mathbf{r}}, \dot{\mathbf{w}} = (\mathbf{A} + \nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{B} + \nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \\ &\dot{\mathbf{y}} = (\mathbf{C} + \nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{D} + \nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\dot{\tilde{\boldsymbol{\rho}}} = \nabla_{\tilde{\mathbf{x}}}\tilde{\boldsymbol{\rho}}\,\tilde{\mathbf{w}} + \nabla_{\mathbf{r}}\tilde{\boldsymbol{\rho}}\,\dot{\mathbf{r}}, \dot{\tilde{\mathbf{w}}} = (\tilde{\mathbf{A}} + \nabla\tilde{\mathbf{f}}(\tilde{\boldsymbol{\rho}})\nabla_{\tilde{\mathbf{x}}}\tilde{\boldsymbol{\rho}})\tilde{\mathbf{w}} + (\tilde{\mathbf{B}} + \nabla\tilde{\mathbf{f}}(\tilde{\boldsymbol{\rho}})\nabla_{\mathbf{r}}\tilde{\boldsymbol{\rho}})\dot{\mathbf{r}} \\ &\dot{\tilde{\mathbf{y}}} = (\tilde{\mathbf{C}} + \nabla\tilde{\mathbf{g}}(\tilde{\boldsymbol{\rho}})\nabla_{\tilde{\mathbf{x}}}\tilde{\boldsymbol{\rho}})\tilde{\mathbf{w}} + (\tilde{\mathbf{D}} + \nabla\tilde{\mathbf{g}}(\tilde{\boldsymbol{\rho}})\nabla_{\mathbf{r}}\tilde{\boldsymbol{\rho}})\dot{\mathbf{r}} \end{aligned} \tag{15}$$

The members of the velocity-based linearisation families are obtained by "freezing" the velocity-based nonlinear systems (14) and (15). Assume that the nonlinear systems are minimal representations in the sense that (1) the members of the velocity-based linearisations families are controllable and observable, (2) the pairs $(\mathbf{A},\mathbf{C})$ and $(\tilde{\mathbf{A}},\tilde{\mathbf{C}})$ are observable, and (3) $\nabla\mathbf{f}(0)$, $\nabla\tilde{\mathbf{f}}(0)$, $\nabla\mathbf{g}(0)$, $\nabla\tilde{\mathbf{g}}(0)$ are equal to zero. Condition 1 is a standard minimality condition from linear theory whilst conditions 2 and 3 remove the possible ambiguity regarding the linear component, if any, of $\mathbf{f}$, $\tilde{\mathbf{f}}$, $\mathbf{g}$, $\tilde{\mathbf{g}}$. Reformulate the velocity-based nonlinear system, (14), as in figure 1; that is, as the nonlinear system

$$\begin{aligned} &\dot{\boldsymbol{\rho}} = \nabla_{\mathbf{r}_{aug}}\boldsymbol{\rho}\,\dot{\mathbf{r}}_{aug}, \dot{\boldsymbol{\omega}} = \mathbf{A}\boldsymbol{\omega} + ([\mathbf{B} \ \ \mathbf{0}] + \nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{r}_{aug}}\boldsymbol{\rho})\dot{\mathbf{r}}_{aug} \\ &\dot{\boldsymbol{\eta}} = \mathbf{C}\boldsymbol{\omega} + ([\mathbf{D} \ \ \mathbf{0}] + \nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{r}_{aug}}\boldsymbol{\rho})\dot{\mathbf{r}}_{aug}, \dot{\boldsymbol{\eta}}_{\rho} = \mathbf{M}\boldsymbol{\omega} + [\mathbf{N} \ \ 0]\,\dot{\mathbf{r}}_{aug} \end{aligned} \tag{16}$$

with augmented input $\dot{\mathbf{r}}_{aug} = \begin{bmatrix}\dot{\mathbf{r}} \\ \dot{\boldsymbol{\rho}}\end{bmatrix}$, enclosed within a unity feedback loop by setting the input, $\dot{\boldsymbol{\rho}}$, equal to the output, $\dot{\boldsymbol{\eta}}_{\rho}$ (where $\mathbf{M} = \nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\mathbf{N} = \nabla_{\mathbf{r}}\boldsymbol{\rho}$ and $\nabla_{\mathbf{r}_{aug}}\boldsymbol{\rho} = [0 \ \ \mathbf{I}]$). Clearly, the nonlinear system, (16), is also in velocity-based form but with the scheduling variable, $\boldsymbol{\rho}$, is now an input. Similarly, reformulate (15) as

$$\begin{aligned} &\dot{\tilde{\boldsymbol{\rho}}} = \nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\boldsymbol{\rho}}\,\dot{\tilde{\mathbf{r}}}_{aug}, \dot{\tilde{\boldsymbol{\omega}}} = \tilde{\mathbf{A}}\tilde{\boldsymbol{\omega}} + ([\tilde{\mathbf{B}} \ \ \mathbf{0}] + \nabla\tilde{\mathbf{f}}(\tilde{\boldsymbol{\rho}})\nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\boldsymbol{\rho}})\dot{\tilde{\mathbf{r}}}_{aug} \\ &\dot{\tilde{\boldsymbol{\eta}}} = \tilde{\mathbf{C}}\tilde{\boldsymbol{\omega}} + ([\tilde{\mathbf{D}} \ \ \mathbf{0}] + \nabla\tilde{\mathbf{g}}(\tilde{\boldsymbol{\rho}})\nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\boldsymbol{\rho}})\dot{\tilde{\mathbf{r}}}_{aug}, \dot{\tilde{\boldsymbol{\eta}}}_{\rho} = \tilde{\mathbf{M}}\tilde{\boldsymbol{\omega}} + [\tilde{\mathbf{N}} \ \ 0]\,\dot{\tilde{\mathbf{r}}}_{aug} \end{aligned} \tag{17}$$

with augmented input $\dot{\tilde{\mathbf{r}}}_{aug} = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\tilde{\rho}} \end{bmatrix}$, enclosed within a unity feedback loop by setting the input, $\dot{\tilde{\rho}}$, equal to the output, $\dot{\tilde{\eta}}_{\rho}$ (where $\tilde{\mathbf{M}} = \nabla_{\tilde{\mathbf{x}}}\tilde{\rho}$, $\tilde{\mathbf{N}} = \nabla_{\mathbf{r}}\tilde{\rho}$ and $\nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\rho} = [\,0 \quad \mathbf{I}\,]$). The corresponding velocity-based linearisation families are obtained, respectively, by "freezing" (16) and (17).

Assume that corresponding members of the velocity-based linearisation families (i.e. for which ρ=ρ_1, $\tilde{\rho}$ =ρ_1) have, respectively, the same transfer function from $\dot{\mathbf{r}}$ to $\begin{bmatrix} \dot{\eta} \\ \dot{\eta}_{\rho} \end{bmatrix}$ and from $\dot{\mathbf{r}}$ to $\begin{bmatrix} \dot{\tilde{\eta}} \\ \dot{\tilde{\eta}}_{\rho} \end{bmatrix}$. It follows immediately from standard linear theory that

$$
\begin{aligned}
&\tilde{\mathbf{A}} = \mathbf{T}(\rho_1)\mathbf{A}\mathbf{T}^{-1}(\rho_1), \quad \left([\tilde{\mathbf{B}} \quad \mathbf{0}] + \nabla\tilde{\mathbf{f}}(\tilde{\rho})\nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\rho}\right) = \mathbf{T}(\rho_1)\left([\mathbf{B} \quad \mathbf{0}] + \nabla\mathbf{f}(\rho)\nabla_{\mathbf{r}_{aug}}\rho\right) \\
&\tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}(\rho_1), \quad \left([\tilde{\mathbf{D}} \quad \mathbf{0}] + \nabla\tilde{\mathbf{g}}(\tilde{\rho})\nabla_{\tilde{\mathbf{r}}_{aug}}\tilde{\rho}\right) = \left([\mathbf{D} \quad \mathbf{0}] + \nabla\mathbf{g}(\rho)\nabla_{\mathbf{r}_{aug}}\rho\right) \\
&\tilde{\mathbf{M}} = \mathbf{M}\mathbf{T}^{-1}(\rho_1), \quad \tilde{\mathbf{N}} = \mathbf{N}
\end{aligned}
\tag{18}
$$

where $\mathbf{T}(\rho_1)$ is a non-singular linear state transformation (which may be different for each member of a linear family). Without loss of generality, let $\mathbf{T}(0)$ be the identity matrix; then owing to the minimality condition 3, it follows that (18) reduces at the origin to $\tilde{\mathbf{A}} = \mathbf{A}$, $\tilde{\mathbf{B}} = \mathbf{B}$, $\tilde{\mathbf{C}} = \mathbf{C}$, $\tilde{\mathbf{D}} = \mathbf{D}$, $\tilde{\mathbf{M}} = \mathbf{M}$, $\tilde{\mathbf{N}} = \mathbf{N}$. Hence, $\mathbf{T}(\rho_1)\mathbf{A}\mathbf{T}^{-1}(\rho_1) = \mathbf{A}, \mathbf{C}\mathbf{T}^{-1}(\rho_1) = \mathbf{C}$ and so

$$
[\mathbf{C} \quad \mathbf{CA} \quad \cdots \quad \mathbf{CA}^{n}]^{T}\mathbf{T}^{-1}(\rho_1) = [\mathbf{C} \quad \mathbf{CA} \quad \cdots \quad \mathbf{CA}^{n}]^{T} \qquad \forall\rho_1 \tag{19}
$$

From the observability of ($\mathbf{A}$, $\mathbf{C}$), the matrix $[\mathbf{C} \quad \mathbf{CA} \quad \cdots \quad \mathbf{CA}^{n}]^{T}$ is full rank and it follows from (19) that $\mathbf{T}(\rho_1)$ must be constant; that is, $\mathbf{T}(\rho_1)$=$\mathbf{T}(0)$=$\mathbf{I}$. Consequently, under the foregoing conditions the nonlinear systems (14) and (15) (and so (12) and (13)) must be identical.

Summary

The reconstruction of a nonlinear dynamic system from a suitable collection of identified linear systems is considered. It is shown that knowledge only of the transfer functions of the linearisations of the nonlinear system is insufficient to permit such reconstruction and sufficient conditions based on augmented transfer function knowledge are derived.

Acknowledgement

D.J.Leith gratefully acknowledges the support provided by the Royal Society for the work presented.

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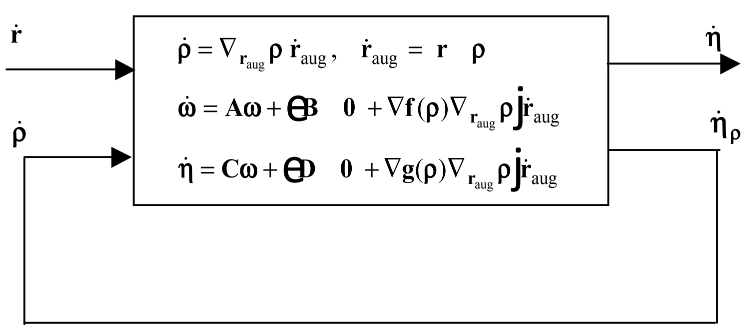

Figure 1 Reformulation to provide additional information from identified transfer function.