# ON LINEAR PARAMETER VARYING FORMULATIONS OF NONLINEAR SYSTEMS

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**ABSTRACT**
A number of recent gain-scheduling approaches assume that the plant to be controlled is in so-called linear parameter-varying form. However, present theory does not support the reformulation of nonlinear systems into linear parameter-varying form without, in general, considerable restrictions either on the class of nonlinear systems considered or on the allowable operating region.. By employing velocity-based linearisation analysis, it is shown that a very general class of nonlinear systems can, indeed, be transformed into linear parameter-varying form.

## 1. INTRODUCTION

Gain-scheduled controllers are linked by the design approach employed, whereby a nonlinear controller is constructed by interpolating, in some manner, between the members of a family of linear time-invariant controllers. In the conventional, and most common, gain-scheduling design approach , each linear controller is typically associated with a specific equilibrium operating point of the plant and is designed to ensure that, locally to the equilibrium operating point, the performance requirements are met. This approach is, essentially, applicable to every nonlinear plant which can be linearised at its equilibrium operating points.

Recently, a number of interesting alternative approaches have been proposed in the context of gain-scheduling design. Since these approaches employ various types of so-called linear parameter-varying (LPV) plant representation, they are commonly referred to as LPV gain-scheduling methods. Unfortunately, these approaches are, on the face of it, restricted to specific classes of plants. Shamma (1988) considers systems $\dot{\mathbf{x}}=\mathbf{A}(\boldsymbol{\theta}(t))\mathbf{x}+\mathbf{B}(\boldsymbol{\theta}(t))\mathbf{r}$, $\mathbf{y}=\mathbf{C}(\boldsymbol{\theta}(t))\mathbf{x}+\mathbf{D}(\boldsymbol{\theta}(t))\mathbf{r}$, where the parameter, $\boldsymbol{\theta}$, is an exogenous time-varying quantity (strictly independent of the state $\mathbf{x}$ of the system) which takes values in some allowable set. Becker *et al.* (1993) consider plants, $\dot{\mathbf{x}}=\mathbf{A}(\boldsymbol{\theta})\mathbf{x}+\mathbf{B}(\boldsymbol{\theta})\mathbf{r}$, $\mathbf{y}=\mathbf{C}(\boldsymbol{\theta})\mathbf{x}+\mathbf{D}(\boldsymbol{\theta})\mathbf{r}$, where the parameter, $\boldsymbol{\theta}$, takes values in some allowable set but may otherwise vary *arbitrarily* with time and so may, for example, depend on $\mathbf{x}$ and $\mathbf{r}$ Similarly, the approaches of Packard (1994), Apkarian & Gahinet (1995), require that the plant is in the form of a linear time-invariant system enclosed by a feedback loop with gains which are bounded but may otherwise vary arbitrarily. More recently, Wu *et al.* (1995), Apkarian & Adams (1997,1998) consider systems where, in addition to requiring that the parameter $\boldsymbol{\theta}$ belongs to some bounded set, it is also assumed that there exists an upper bound on the rate of variation of $\boldsymbol{\theta}$. This work clearly begs the question of whether the class of plants considered is, indeed, sufficiently rich to include a class of nonlinear plants which is of comparable generality to alternative control design methods such as the conventional gain-scheduling approach. Indeed, on the face of it few nonlinear systems have dynamics which are of the required LPV form. The purpose of this note is, therefore, to investigate the prevalence of nonlinear systems with LPV dynamics.

## 2. LPV REPRESENTATIONS

In the literature, the term "linear parameter-varying" is widely employed to refer to any system of the form

$$\dot{\mathbf{x}}=\mathbf{A}(\boldsymbol{\theta})\mathbf{x}+\mathbf{B}(\boldsymbol{\theta})\mathbf{r},\ \ \mathbf{y}=\mathbf{C}(\boldsymbol{\theta})\mathbf{x}+\mathbf{D}(\boldsymbol{\theta})\mathbf{r} \tag{1}$$

where $\boldsymbol{\theta}$ is a parameter belonging to some class $\Omega$. However, although superficially similar, it is emphasised that the dynamic characteristics of systems, (1), are strongly dependent on the class $\Omega$ to which the parameters belong. In particular, when $\boldsymbol{\theta}$ is permitted to depend on the state, $\mathbf{x}$[1], (the situation studied, e.g., by Apkarian *et al.* 1995, Scherer *et al.* 1997, Apkarian & Adams 1998) the dependence of the **A** and **B** matrices on the state introduces nonlinear feedback not present in linear time-invariant/time-varying systems. Use of the term "linear parameter-varying" to describe such nonlinear systems is, therefore, potentially misleading. Consequently, in the context of the present paper, a generalisation of the terminology of Shamma (1988) is adopted and systems, (1), where $\boldsymbol{\theta}$ may depend on the state,

[1] It should be noted that it is usually necessary to restrict the input and initial conditions of the state such that the solution $\mathbf{x}(t)$ is confined to some bounded operating region $X \subset \Re^n$ thereby ensuring that the parameter $\boldsymbol{\theta}$ is bounded.

$\mathbf{x}$, are hereafter referred to as quasi-LPV systems while the term LPV is reserved for systems where $\boldsymbol{\theta}$ is a strictly exogenous time-varying quantity (strictly independent of the state $\mathbf{x}$ of the system).

### 2.1 LPV Systems

Perhaps the earliest work on LPV systems in the context of gain-scheduling is that of Shamma (1988). Shamma (1988), considers systems

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\theta}(t))\mathbf{x}+\mathbf{B}(\boldsymbol{\theta}(t))\mathbf{r}, \ \mathbf{y}=\mathbf{C}(\boldsymbol{\theta}(t))\mathbf{x}+\mathbf{D}(\boldsymbol{\theta}(t))\mathbf{r} \quad (2)$$

where the parameter, $\boldsymbol{\theta}$, is an exogenous time-varying quantity which takes values in some allowable set. Under these conditions, an LPV system is simply a particular form of linear time-varying system. Linear time-varying representations of nonlinear systems are largely associated with series expansion linearisation theory and this is, therefore, briefly reviewed in this section.

*2.1.1 Series expansion linearisation theory*

Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \ \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \quad (3)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable with bounded, Lipschitz continuous derivatives. The set of equilibrium operating points of the nonlinear system, (3), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which $\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0$. Let $\Phi:\Re^n\times\Re^m$ denote the space consisting of the union of the state, $\mathbf{x}$, with the input, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear system, (3), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in $\Phi$ and the response of the system to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in $\Phi$. Let $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$ denote a specific trajectory of the nonlinear system, (3); that is, $\dot{\tilde{\mathbf{x}}} = \mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$, $\tilde{\mathbf{y}} = \mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$. The trajectory, $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$, could be an equilibrium operating point of (3), in which case $\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$ is identically zero and $\tilde{\mathbf{x}}$ is a constant. The nonlinear system, (3), may be reformulated, relative to the trajectory $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$, as,

$$\delta\dot{\mathbf{x}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} + \boldsymbol{\varepsilon}_\mathbf{F} \quad (4)$$

$$\delta\mathbf{y} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} + \boldsymbol{\varepsilon}_\mathbf{G} \quad (5)$$

$$\delta\mathbf{r} = \mathbf{r} - \tilde{\mathbf{r}}, \ \mathbf{y} = \delta\mathbf{y} + \tilde{\mathbf{y}}, \ \mathbf{x} = \delta\mathbf{x} + \tilde{\mathbf{x}} \quad (6)$$

where $\boldsymbol{\varepsilon}_\mathbf{F}$, $\boldsymbol{\varepsilon}_\mathbf{G}$ are residual terms. From Taylor series expansion theory

$$|\boldsymbol{\varepsilon}_\mathbf{F}| \le \sigma(|\delta\mathbf{x}| + |\delta\mathbf{r}|)^2, \quad |\boldsymbol{\varepsilon}_\mathbf{G}| \le \sigma(|\delta\mathbf{x}| + |\delta\mathbf{r}|)^2 \quad (7)$$

where $\sigma$ is a finite positive constant (e.g. Desoer & Vidyasagar 1975 p130). Hence, provided $|\delta\mathbf{x}|$ and $|\delta\mathbf{r}|$ are sufficiently small, the dynamics, (4)-(5), can be approximated by the linear time-varying system

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\mathbf{r} \quad (8)$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\mathbf{r} \quad (9)$$

The system, (8)-(9), is in LPV form (the parameter is an exogenous time-varying quantity independent of the state) and is simply the first-order Taylor series expansion of the nonlinear system, (3), relative to the trajectory, $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$. The system, (8)-(9), approximates (4)-(5), in the sense that the solution to (8)-(9) approximates the solution to (4)-(5) (e.g., Khalil 1992 theorem 2.5). Moreover, (4)-(5) is exponentially stable, locally to the trajectory $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$, if and only if (8)-(9) is stable (e.g. Khalil 1992 theorem 4.6). Since consideration is confined to a specific trajectory, $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t), \tilde{\mathbf{y}}(t))$, the transformations, (6), are static. Hence, the solution to the nonlinear system, (3), is directly related to the solution to (4)-(5).

*2.2.2 Families of series expansion linearisations*

Since the series expansion linearisation about an equilibrium operating point is only valid in a small operating region about that point, it might be expected that the allowable operating region may be enlarged by combining, in some sense, the series expansion linearisations about a number of equilibrium operating points. That is, it might be expected that when the solution to the nonlinear system (3) remains within a neighbourhood about the locus of equilibrium operating points (but is no longer confined to the vicinity of a single equilibrium point), it is directly related to the solutions to the members of the series expansion linearisation family

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\delta\mathbf{r} \quad (10)$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\delta\mathbf{r} \quad (11)$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}, \ \hat{\mathbf{y}} = \delta\hat{\mathbf{y}} + \mathbf{G}(\mathbf{x_o}, \mathbf{r_o}), \ \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x_o} \quad (12)$$

where $(\mathbf{x_o}, \mathbf{r_o})$ is an equilibrium operating point of the nonlinear system, (3).

However, it is important to make a clear distinction between the LPV system, (1), and the family of linear systems associated with the equilibrium points/trajectories of a nonlinear system, (3). Clearly, the linearisation family is a collection of dynamic systems whilst the LPV system is a single dynamic system. The state, input and output of a series expansion linearisation are perturbation quantities which depend on the equilibrium point/trajectory considered. Hence, the members of the linearisation family each have different state, input and output in general and when the solution to (3) traces a trajectory which is not confined to a neighbourhood about a single equilibrium operating point, the relationship between the solution to the nonlinear system, (3), and the solutions to the linear systems, (10)-(11), is, in fact, no longer straightforward. It is emphasised that the input, output and state transformations, (12), are essential to the relationship between the nonlinear system and its series expansion linearisations and cannot be neglected. Taking account of the input, output and state transformations, the members of the series expansion linearisation family, (10)-(12), may be reformulated as,

$$\dot{\hat{\mathbf{x}}} = -\{\nabla_x \mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{x}_o + \nabla_r \mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{r}_o\} + \nabla_x \mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\,\hat{\mathbf{x}} + \nabla_r \mathbf{F}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{r} \quad (13)$$

$$\hat{\mathbf{y}} = \{\mathbf{G}(\mathbf{x}_o,\mathbf{r}_o) - \nabla_x \mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{x}_o - \nabla_r \mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{r}_o\} + \nabla_x \mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\,\hat{\mathbf{x}} + \nabla_r \mathbf{G}(\mathbf{x}_o,\mathbf{r}_o)\mathbf{r} \quad (14)$$

The members, of the family of first-order representations, (13)-(14), have the same input, output and state. The solutions to the members of this family can be combined to approximate the solution to the nonlinear system, (3), in a neighbourhood about the locus of equilibrium operating points (Leith & Leithead 1998a). However, the members of the family, (13)-(14), are affine rather than linear. It is emphasised that affine systems do not satisfy superposition and, since the inhomogeneous term may be very large, cannot generally be treated as linear systems subject to a small disturbance. The corresponding combined representation is neither an LPV system nor a quasi-LPV system.

## 2.2 Quasi-LPV systems

Conventional series expansion linearisation theory does not support the reformulation of general nonlinear systems in LPV form without strong restrictions on the operating region. However, following a similar approach to Helmersson (1995 chapter 10) (see also Scherer *et al.* (1997), Apkarian & Adams 1998 section IV), consider the nonlinear quasi-LPV system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x},\mathbf{r})\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{r})\mathbf{r}, \quad \mathbf{y} = \mathbf{C}(\mathbf{x},\mathbf{r})\mathbf{x} + \mathbf{D}(\mathbf{x},\mathbf{r})\mathbf{r} \qquad (15)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$ and the input and initial conditions of the state are restricted such that $\mathbf{r} \in R \subset \Re^m$ and the solution $\mathbf{x}(t)$ is confined to some operating region $X \subset \Re^n$. It is immediately evident that the solutions to the nonlinear system, (15), are a subset of the solutions to the system, (1), with $\boldsymbol{\theta} \in X \times R$. (Since the parameter $\boldsymbol{\theta}$ can vary arbitrarily in (1), the solutions to (15) are just the solutions to (1) associated with particular parameter trajectories). Hence, whilst it is generally not possible to reformulate a nonlinear system as an LPV system, it is possible to over-bound the general nonlinear system, (15), by a quasi-LPV system, (15), in the sense that every solution to the nonlinear system is a solution to the quasi-LPV system (but not *vice versa*). Of course, some degree of conservativeness can be expected with such an approach. Moreover, it still remains to be established whether the class of quasi-LPV systems is sufficiently rich to include a reasonable range of gain-scheduling applications. The pseudo-linear theory and output-dependent quasi-LPV theory relating to the reformulation of nonlinear systems in quasi-LPV form are reviewed in the following sections.

### *2.2.1Higher-order series expansions*

The series expansion linearisation of a nonlinear system is obtained by truncating the Taylor series expansion of $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ after the first two terms. Owing to this truncation, the series expansion linearisation representation is only valid locally to a specific trajectory or equilibrium operating point. Provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable sufficiently many times, the neighbourhood within which the series expansion representation is valid can be enlarged by truncating the series expansion later thereby retaining more higher-order terms. For example, Banks & Al-Jurani (1996) consider the unforced nonlinear system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ (16) where $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot)$ is analytic, and $\mathbf{F}(0) = 0$. By retaining infinitely many higher-order terms in the Taylor series expansion of $\mathbf{F}(\cdot)$ about the origin, (16) may be reformulated as the quasi-LPV system $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x}$ provided the initial condition of $\mathbf{x}$ is restricted such that the components of $\mathbf{x}(t)$ are uniformly bounded. Whilst Banks & Al-Jurani (1996) restrict consideration to unforced systems, the extension to forced systems, (4), is straightforward provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic and $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ (e.g. Helmersson 1995 chapter 10). The assumption that $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ is not restrictive, provided the system has at least one equilibrium operating point, since this requirement can always be satisfied by adding/subtracting a constant from the state, input and output. The requirement that $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic (and therefore infinitely differentiable) is rather stronger. Nevertheless, the main difficulties with this approach are essentially practical in nature. Owing to the difficulty of evaluating the higher-order derivatives of a non-linear function and the sensitivity of polynomial series expansions to errors in the coefficients of the higher order terms, the infinite series approach of Banks & Al Jurani (1996) is impractical and it is almost always necessary, in practice, to employ a truncated series with only a finite number of terms. Indeed, it is frequently necessary to truncate the infinite series after only a relatively small number of terms, particularly for high-order systems with a large number of states, inputs and outputs. Hence, the quasi-LPV representation obtained by this method is, in practice, only valid within a neighbourhood about the origin. Whilst this neighbourhood subsumes that within which the series expansion linearisation is valid, it may nevertheless still be small.

### *2.2.2 Reformulation by mean value theorem*

In an approach which is closely related to that considered in the previous section, the mean value theorem can be employed to reformulate a general nonlinear system in quasi-LPV form. Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}) \qquad (17)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ is differentiable with bounded, Lipschitz continuous derivatives. It follows from the mean value theorem (e.g. Boyd *et al.* 1994) that

$$\mathbf{c}^T\left(\mathbf{F}(\mathbf{x},\mathbf{r}) - \mathbf{F}(\bar{\mathbf{x}},\bar{\mathbf{r}})\right) = \mathbf{c}^T\nabla_x\mathbf{F}(\mathbf{z}_x,\mathbf{z}_r)\left(\mathbf{x} - \bar{\mathbf{x}}\right) + \mathbf{c}^T\nabla_r\mathbf{F}(\mathbf{z}_x,\mathbf{z}_r)\left(\mathbf{r} - \bar{\mathbf{r}}\right) \qquad (18)$$

where $\mathbf{c}^T \in \Re^n$ and $(\mathbf{z}_x,\mathbf{z}_r)$ is a point lying on the line segment in $\Re^n \times \Re^m$ joining $(\mathbf{x},\mathbf{r})$ and $(\bar{\mathbf{x}},\bar{\mathbf{r}})$. Hence, assuming without loss of generality that $\mathbf{F}(0,0)=0$, then

$$\mathbf{c}^T\mathbf{F}(\mathbf{x},\mathbf{r}) = \mathbf{c}^T\nabla_x\mathbf{F}(\mathbf{z}_x,\mathbf{z}_r)\mathbf{x} + \mathbf{c}^T\nabla_r\mathbf{F}(\mathbf{z}_x,\mathbf{z}_r)\mathbf{r} \qquad (19)$$

Since $\mathbf{c}$ can take any value in $\Re^n$, it follows that

$$\dot{\mathbf{x}} = \begin{bmatrix} [\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_1}, \mathbf{z}_{\mathbf{r}_1})]_1 \\ \vdots \\ [\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_n}, \mathbf{z}_{\mathbf{r}_n})]_n \end{bmatrix} + \begin{bmatrix} [\nabla_{\mathbf{r}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_1}, \mathbf{z}_{\mathbf{r}_1})]_1 \\ \vdots \\ [\nabla_{\mathbf{r}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_n}, \mathbf{z}_{\mathbf{r}_n})]_n \end{bmatrix} \quad (20)$$

where $[\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})]_i$ denotes the i$^{th}$ row of $\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$ and the $(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$, i=1,..n, are points which lie on the line segment in $\Re^n \times \Re^m$ from (**x**,**r**) to the origin. It should be noted that the points $(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$, i=1,..n, strongly depend, in general, on the value of (**x**,**r**) and so vary as the solution (**x**(t),**r**(t)) to the system evolves. Evidently, the dynamics, (20), are in quasi-LPV form with parameter $\boldsymbol{\theta} = [\mathbf{z}_{\mathbf{x}_1}{}^T \ \mathbf{z}_{\mathbf{r}_1}{}^T \ \cdots \ \mathbf{z}_{\mathbf{x}_n}{}^T \ \mathbf{z}_{\mathbf{r}_n}{}^T]^T$.

Unfortunately, whilst applicable to a general class of systems, the reformulation, (20), is essentially an existence result and, in particular, provides little insight regarding a general means for determining the specific value of $\boldsymbol{\theta}$ associated with an operating point, (**x**,**r**). LPV control synthesis techniques require that a measurement or estimate of the varying parameter, $\boldsymbol{\theta}$, is available to the controller. In addition, owing to the conservativeness of controllers designed to accommodate arbitrary rates of parameter variation, it is often required that an upper bound can be placed on $\dot{\boldsymbol{\theta}}$. Hence, it is necessary to explicitly establish a mapping from (**x**,**r**) to $\boldsymbol{\theta}$; that is, from (**x**,**r**) to $[\mathbf{z}_{\mathbf{x}_1}{}^T \ \mathbf{z}_{\mathbf{r}_1}{}^T \ \cdots \ \mathbf{z}_{\mathbf{x}_n}{}^T \ \mathbf{z}_{\mathbf{r}_n}{}^T]^T$. This is highly non-trivial in general, particularly for high-order multivariable systems, which greatly diminishes the utility of the formulation, (20). Moreover, the interpretation of the linear system obtained by "freezing" the parameter $\boldsymbol{\theta}$ in (20) at a particular value is unclear in the sense that no direct relationship exists between the solution to the frozen-parameter linear system and the solution to the nonlinear quasi-LPV system. Indeed, the frozen-parameter system associated with an equilibrium point is quite different from the conventional series expansion linearisation at that point.

### *2.2.3 Output dependent quasi-LPV systems*

LPV synthesis techniques require that a measurement or estimate of the varying parameter, $\boldsymbol{\theta}$, is available and this naturally leads to consideration of systems where $\boldsymbol{\theta}$ is one of the outputs. In particular, Shamma (1988) considers nonlinear systems

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{y}) \\ \mathbf{f}_{\mathbf{x}}(\mathbf{y}) \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{\mathbf{yy}}(\mathbf{y}) & \mathbf{A}_{\mathbf{yx}}(\mathbf{y}) \\ \mathbf{A}_{\mathbf{xy}}(\mathbf{y}) & \mathbf{A}_{\mathbf{xx}}(\mathbf{y}) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{\mathbf{y}} \\ \mathbf{B}_{\mathbf{x}} \end{bmatrix} \mathbf{r} \quad (21)$$

where $\mathbf{y} \in \Re^m$, $\mathbf{x} \in \Re^{n-m}$ and the input, $\mathbf{r} \in \Re^m$, has the same dimension as the subset of state, **y**. It is assumed that **r** is uniformly zero at every equilibrium operating point and, in addition, it is assumed that the family of equilibrium operating points are smoothly parameterised by **y**. Under these conditions, (21) may be transformed into the quasi-LPV system

$$\dot{\boldsymbol{\xi}} = \mathbf{A}(\mathbf{y})\boldsymbol{\xi} + \mathbf{B}(\mathbf{y})\mathbf{r} \quad (22)$$

where the matrices **A**(**y**) and **B**(**y**) are appropriately defined, $\boldsymbol{\xi}$ equals $[\mathbf{y} \ \ \mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})]^T$ and $\mathbf{x_o}(\mathbf{y})$ is the equilibrium value of **x** corresponding to **y** (Shamma & Athans 1992).

Of course, few systems are of the form, (21): in particular, it is restrictive to require that the nonlinearity is dependent only on the quantity, **y**, which parameterises the equilibrium operating points and that the input is uniformly zero at every equilibrium operating point. However, with regard to the former restriction, consider the nonlinear system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, \mathbf{r}) \quad (23)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{z} = [\mathbf{y} \ \ \mathbf{x}]^T$ with $\mathbf{y} \in \Re^m$, $\mathbf{x} \in \Re^{n-m}$ and $\mathbf{F}(\cdot,\cdot) = [\mathbf{F_y}(\cdot,\cdot) \ \ \mathbf{F_x}(\cdot,\cdot)]^T$ is differentiable with bounded, Lipschitz continuous derivatives. (It should be noted that, when the output function $\mathbf{G}(\cdot,\cdot)$ does not depend on the input **r**, the nonlinear system, (4), can be reformulated as in (1)). Assume that the family of equilibrium operating points are smoothly parameterised by **y**. Adopting a similar approach to Shamma & Athans (1992) and employing a partial first-order series expansion about the equilibrium operating point, $(\mathbf{x_o}(\mathbf{y}),\mathbf{r_o}(\mathbf{y}))$, the nonlinear system may be reformulated as

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \delta\dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 0 & \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) \\ 0 & \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{x}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) - \nabla \mathbf{x_0}(\mathbf{y}) \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \delta\mathbf{x} \end{bmatrix} \quad (24)$$

$$+ \begin{bmatrix} \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) \\ \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{x}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) - \nabla \mathbf{x_0}(\mathbf{y}) \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, \mathbf{r_0}(\mathbf{y})) \end{bmatrix} \delta\mathbf{r} + \begin{bmatrix} \boldsymbol{\varepsilon}_{\mathbf{y}} \\ \boldsymbol{\varepsilon}_{\mathbf{x}} \end{bmatrix}$$

$$\delta\mathbf{x} = \mathbf{x} - \mathbf{x_o}(\mathbf{y}), \ \delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}(\mathbf{y}) \quad (25)$$

where $\boldsymbol{\varepsilon}_{\mathbf{y}}$, $\boldsymbol{\varepsilon}_{\mathbf{x}}$ are residual terms. Assume that **r** is uniformly zero at every equilibrium operating point; that is, $\mathbf{r_o}(\mathbf{y})=0$ and $\delta\mathbf{r} = \mathbf{r}$. It follows that

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \delta\dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 0 & \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) \\ 0 & \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{x}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) - \nabla \mathbf{x_0}(\mathbf{y}) \nabla_{\mathbf{x}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \delta\mathbf{x} \end{bmatrix} \quad (26)$$

$$+ \begin{bmatrix} \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) \\ \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{x}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) - \nabla \mathbf{x_0}(\mathbf{y}) \nabla_{\mathbf{r}} \mathbf{F}_{\mathbf{y}}([\mathbf{y} \ \ \mathbf{x_0}(\mathbf{y})]^T, 0) \end{bmatrix} \mathbf{r} + \begin{bmatrix} \boldsymbol{\varepsilon}_{\mathbf{y}} \\ \boldsymbol{\varepsilon}_{\mathbf{x}} \end{bmatrix}$$

Neglecting the perturbation terms $\boldsymbol{\varepsilon}_{\mathbf{y}}$ and $\boldsymbol{\varepsilon}_{\mathbf{x}}$, it is evident that (1) is of the quasi-LPV form (22). From Taylor series expansion theory, the perturbation terms $\boldsymbol{\varepsilon}_{\mathbf{y}}$ and $\boldsymbol{\varepsilon}_{\mathbf{x}}$ can be made arbitrarily small provided the magnitudes of **r** and $\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})$ are sufficiently small. Hence, provided |**r**| and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ are sufficiently small, the solution to the nonlinear system, (1), is approximated by the solution to quasi-LPV system obtained by neglecting $\boldsymbol{\varepsilon}_{\mathbf{y}}$ and $\boldsymbol{\varepsilon}_{\mathbf{x}}$ in (1). It should be noted that the matrices, **A**(**y**) and **B**(**y**), in the quasi-LPV system do *not* correspond to the series expansion linearisations, about the family of equilibrium operating points, of the nonlinear system, (1).

The foregoing analysis relaxes the requirement that the nonlinearity is dependent solely on the quantity, **y**, which parameterises the equilibrium operating points. However, this is achieved at the cost of restricting consideration to the class of inputs and initial conditions for which $|\mathbf{r}|$ and $|\mathbf{x}-\mathbf{x_o}(\mathbf{y})|$ are sufficiently small. Hence, any analysis based on this quasi-LPV formulation is strictly local in nature. It should be noted that, from (1), restricting $|\mathbf{r}|$ and $|\mathbf{x}-\mathbf{x_o}(\mathbf{y})|$ imposes an implicit constraint on the rate of variation of the states, $\mathbf{y}$ and $\mathbf{x}$. This constraint may, in general, be extremely restrictive since the region of validity can be vanishingly small.

Moreover, the requirement that $\mathbf{r}$ is uniformly zero at the equilibrium operating points is still necessary since it ensures that the input transformation, (25), associated with the series expansion is trivial and, in particular, does not vary with the equilibrium operating point. Indeed, this condition is central to both the approach of Shamma & Athans (1992) and to the foregoing quasi-LPV reformulation. (When this condition is not satisfied, the quasi-LPV approximation is only accurate about the specific equilibrium operating point at which $\mathbf{r_o}$ is zero). In the specific control design situation where the nonlinear system to be reformulated is the plant and the controller output is the only input to the plant and the controller contains pure integral action, this condition can be satisfied by formally including the controller integral action within the plant so that the input, $\mathbf{r}$, to the augmented plant is zero in equilibrium. It should be noted, however, that in a more general context the requirement that the input is uniformly zero in equilibrium is rather restrictive; for example, when the input, $\mathbf{r}$, consists of command signals and/or disturbances. It should be noted that the requirement in the foregoing analysis that there exists a sub-set, $\mathbf{y}$, of the state which parameterises the equilibrium points can be relaxed. However, the quasi-LPV system obtained is then only valid in the vicinity of a *single* equilibrium point.

## 3. VELOCITY-BASED REPRESENTATIONS

Nonlinear systems of the particular form, (21), and nonlinear systems belonging to the specific class for which the infinite series expansion can be expressed in closed-form, may be reformulated as quasi-LPV systems. However, these classes of nonlinear system are rather restrictive in comparison to those permitted in other control design approaches. Representations of more general nonlinear systems, (3), in either LPV or quasi-LPV form are strictly local in nature and confined to a small neighbourhood about a specific trajectory, equilibrium operating point or family of equilibrium operating points. Hence, present theory does *not* support the representation of general nonlinear dynamic systems in LPV or quasi-LPV form without, in general, considerable restrictions either on the class of nonlinear systems considered or on the allowable operating region. The requirement is to determine whether these restrictions can be relaxed.

Consider the reformulation of the nonlinear system, (3), in quasi-LPV form. The restrictions on the operating region essentially arise from the limitations of series expansion linearisation theory. In particular, the conventional series expansion of a nonlinear system is, in general, linear only when the expansion is carried out relative to a specific equilibrium operating point (or trajectory). Recently, Leith & Leithead (1998a,b) have developed a velocity-based analysis framework which associates a linear system with every operating point of a nonlinear system, rather than just the equilibrium operating points. In the present context, the velocity-based linearisation approach therefore clearly has the potential to provide insight the representation of nonlinear systems in quasi-LPV form.

Suppose that the nonlinear system, (3), is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, and at time, $t_1$, the trajectory has reached the point, $(\mathbf{x_1}, \mathbf{r_1})$. In the vicinity of the operating point, $(\mathbf{x_1}, \mathbf{r_1})$, the solution, $\mathbf{x}(t)$, to the nonlinear system is approximated (Leith & Leithead 1998a) by the solution, $\hat{\mathbf{x}}(t)$, to the linear system

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}}, \quad \dot{\hat{\mathbf{w}}} = \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{F}(\mathbf{x_1},\mathbf{r_1})\,\dot{\mathbf{r}} \tag{27}$$

$$\dot{\hat{\mathbf{y}}} = \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \tag{28}$$

There exists a velocity-based linearisation, (27)-(28), for every operating point. It follows that a velocity-based linearisation family, with members defined by (27)-(28), can be associated with the nonlinear system, (3). Whilst the solution to an individual velocity-based linearisation is only accurate in the vicinity of the corresponding operating point, the solutions to the members of the velocity-based linearisation family can be pieced together to globally approximate, to an arbitrary degree of accuracy, the solution to the nonlinear system (Leith & Leithead 1998a). Hence, the velocity-based linearisation family embodies the entire dynamics of the nonlinear system, (3), with no loss of information and provides an alternative representation of the nonlinear system. The relationship between the nonlinear system, (3), and its associated velocity-based linearisation family is direct. Differentiating (3), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{29}$$

$$\dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{30}$$

Clearly, the velocity-based linearisation, (27)-(28), is simply the frozen form of (29)-(30) at the operating point, $(\mathbf{x_1}, \mathbf{r_1})$. It should be noted that the differentiation step here is purely formal in nature and does not require differentiation of noisy measurements. The velocity representation, (29)-(30), may be reformulated as the quasi-LPV system

$$\dot{\mathbf{x}} = \mathbf{w}, \quad \dot{\mathbf{r}} = \mathbf{z}, \quad \dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{z} \tag{31}$$

$$\dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{z} \tag{32}$$

where $\mathbf{z}$ is the input to the transformed system. Hence, it follows immediately that every nonlinear system, (3), can be reformulated as an quasi-LPV system, (31)-(32)[2]. In

[2] Of course, a number of issues relating to this reformulation must be addressed, including the realisation of the input

contrast to previous approaches, the reformulation is valid for a very general class of nonlinear systems. Moreover, it is emphasised that the quasi-LPV representation is valid globally with no restriction whatsoever to a neighbourhood of the equilibrium operating points.

## 4. CONCLUSION

The study of LPV gain-scheduling methods is currently the subject of great interest in the literature. These methods are related by the use of various types of linear parameter-varying representation. However, on the face of it, few nonlinear systems are of the required form. While quite a number of approaches have been proposed in the literature for transforming a nonlinear system into a suitable linear parameter-varying form, it is noted here that the results obtained do *not* support the representation of nonlinear dynamic systems in linear parameter-varying form without, in general, considerable restrictions either on the class of nonlinear systems considered or on the allowable operating region.

Employing the velocity-based framework of Leith & Leithead (1998a), it is shown in this note that every smooth nonlinear system of the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$ can indeed be transformed into linear parameter-varying form. In contrast to previous approaches, the reformulation is valid for a very general class of nonlinear systems and it is emphasised that the linear parameter-varying representation is valid globally with no restriction whatsoever to a neighbourhood of the equilibrium operating points. The velocity-based formulation therefore enables LPV gain-scheduling methods to be applied to a very general class of nonlinear systems.

Furthermore, an important aspect of the velocity-based quasi-LPV formulation in the context of gain-scheduling is that a direct relationship exists between the nonlinear quasi-LPV systems and the linear system obtained by simply "freezing" the system at a particular parameter value. Specifically, the solution to the linear system approximates, to second order, the solution to the nonlinear system locally to the operating point associated with the parameter value considered. This relationship is valid at *every* operating point, including those far from equilibrium, not just the equilibrium points and provides a rigorous basis for inferring the dynamic characteristics of velocity-based quasi-LPV system from those of its associated family of frozen-parameter linear systems.

The utility of velocity-based methods is not confined to LPV gain-scheduling approaches but also extends, for example, to other gain-scheduling approaches. By associating a family of linear systems with a nonlinear system while avoiding any constraint to near equilibrium operation, velocity-based methods enable many of the restrictions inherent to conventional gain-scheduling methods to be relaxed whilst maintaining an open design framework and a divide and conquer methodology (see, for example, Leith & Leithead 1998b, 1999).

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transformation employed and the increased order of the velocity-based quasi-LPV representation compared to the direct representation, (3). These issues, and others, which are outwith the scope of the present note, are discussed in detail in Leith & Leithead (1998a,b)


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