Stability analysis and proportional PDC design of the Takagi-Sugeno (T-S) model fuzzy system

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Abstract: **-** This paper presents stability analysis and a novel approach of designing Takagi-Sugeno fuzzy control systems. It is shown that by using the new approach, termed Proportional Parallel Distributed Compensation (PPDC), controller parameters can be significantly reduced. Furthermore, based on the Lyapunov stability analysis, the process of finding the common positive definite matrix P will be simplified and proportional coefficients design can be separated from feedback matrix parameters. The new approach has promising potential in practical applications of fuzzy controllers.

Key-Words: Stability, Takagi-Sugeno fuzzy control, Parallel Distributed Compensation.

1 Introduction

Since the seminal work by Zadeh, fuzzy logic control has become a very popular research area in control engineering. Recently, some analytical tools concerning stability of a closed-loop control system and design of fuzzy controllers had been presented. Most of these research are based on stability analysis and systematic controller design using the Takagi-Sugeno fuzzy model.

Based on this model, a conceptually simple and straightforward algorithm termed parallel distributed compensation (PDC) for designing the controller in order to guarantee the stability of the closed-loop system has been introduced. Recently, Tanaka suggested the idea of using Linear Matrix Inequality (LMI) to find the common P matrix in order to guarantee stability of a T-S fuzzy system [1]. Furthermore, researchers have become increasingly interested in improving and simplifying stability analysis of a T-S fuzzy model and controller design. Some researchers proposed the simplified linear control rules to drastically reduce the number of parameters while maintaining the spirit and advantages of the original T-S fuzzy model [2]. Other presented different approaches of PDC design [3], while some revised the Lyapunov's conditions to improve stability analysis and design of fuzzy control systems [4]. However, these methods are applicable to low-order systems only. Because of the difficulty in analysing high-order systems, more work need to be done in this field.

In this paper, we present a proportional PDC control design approach both for CFS (Continuous Fuzzy System) and DFS (Discrete Fuzzy System). It turns out that the use of proportional coefficients could reduce the adjustable parameters involved in the normal PDC approach greatly and separate them from the feedback gain. Through Lyapunov stability analysis, we derive new theorems concerning sufficient condition to ensure stability. By applying LMI tools, the process of finding the common positive definite matrix P is also simplified. Finally, to illustrate the merit of our approach, numerical simulation studies of stabilising a mass-spring-damper system will be given.

2 Takagi-Sugeno Fuzzy Model

This fuzzy model is composed of input fuzzy sets, fuzzy logic AND operators, fuzzy rules with linear functions of input variables, and a defuzzifier. The ith rule of the T-S fuzzy model is of the following form.

CFS: Rule i:
IF *$z_1(t)$ is M_{i1} AND …AND $z_p(t)$ is M_{ip}*
Then $\dot{x}(t) = A_i x(t) + B_i u(t)$ (1)
i=1, 2,… , r.

DFS: Rule i:
IF *$z_1(t)$ is M_{i1} AND …AND $z_p(t)$ is M_{ip}*
Then $x(t+1) = A_i x(t) + B_i u(t)$ (2)
i=1,2,…,r.

where M_{ij} is the fuzzy set and r is the number of IF-THEN rules, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input vector, $A_i \in R^{n \times n}$ and $B_i \in R^{n \times m}$. $z_1(t)$~$z_p(t)$ are known as premise variables. Each linear state and the output equation in the consequent part is called a "subsystem".

Given a pair of (*x(t) u(t)*), the overall fuzzy system is inferred by using the Centre of Gravity method for defuzzification. They are as follows:

CFS:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(z(t))\{A_i x(t) + B_i u(t)\}}{\sum_{i=1}^{r} w_i(z(t))} \tag{3}$$

$$= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\}$$

DFS:

$$x(t+1) = \frac{\sum_{i=1}^{r} w_i(z(t))\{A_i x(t) + B_i u(t)\}}{\sum_{i=1}^{r} w_i(z(t))} \tag{4}$$

$$= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\}$$

where

$$z(t) = [z_1(t) \quad z_2(t) \ldots z_p(t)]^T$$

$$w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t)), \quad \begin{cases} \sum_{i=1}^{r} w_i(z(t)) > 0 \\ w_i(z(t)) \geq 0 \end{cases}$$

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}, \quad \begin{cases} \sum_{i=1}^{r} h_i(z(t)) = 1 \\ h_i(z(t)) \geq 0 \end{cases}$$

i=*1, 2, …r.*

and $M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in the fuzzy set M_{ij}. Combining membership values of the fuzzy sets in the rule antecedent, the product fuzzy logic AND operator is used here to obtain $w_i(z(t))$ (other AND operators also can be used). $h_i(z(t))$ is denoted as the normalised weight of each fuzzy rule. For the convenience of notation, let $w_i = w_i(z(t))$ and $h_i = h_i(z(t))$. Then, the final state of the fuzzy system can be represented as:

CFS:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\{A_i x(t) + B_i u(t)\} \tag{5}$$

DFS:

$$x(t+1)=\sum_{i=1}^{r} h_i \{A_i x(t)+B_i u(t)\} \qquad (6)$$

Assumption 1: P input variables, and $z_1 \sim z_p$, are described in the T-S fuzzy model. It is assumed that each input variable was fuzzified by two fuzzy sets. Hence, the total number of possible fuzzy rules is 2^p.

3 Proportional PDC Design

PDC uses the technique of feedback stabilisation in the control design. The resulting controller is non-linear and time varying in nature.

The idea is to design a compensator for each rule of the fuzzy model. The designed fuzzy controller shares the same input variables and fuzzy sets with the T-S fuzzy model in the IF parts. For the fuzzy models (1) and (2), the following controller rules are designed via PDC:

Rule i:
IF *$z_1(t)$ is M_{i1} AND …AND $z_p(t)$ is M_{ip}*
Then *$u_i(t)=-F_i x(t)$*
$i=1,2,\dots,r.$

where $F_i \in R^{m \times n}$ is the local state feedback matrix. Each fuzzy control rule has a linear state feedback law in the THEN parts.

In the original PDC design, $r \times m \times n$ adjustable controller parameters empower the controller to generate desired control actions. However, compared with the widely used PI, PD and PID controllers which require tuning of only two or three parameters, the T-S controller using PDC is extremely difficult to use. As a matter of fact, when a human operator tunes the controller parameters, tuning more than 10 parameters simultaneously is proven to be almost impossible. To overcome this drawback, a control rule scheme, which can significantly reduce the number of parameters in PDC, is proposed:

Rule i:
IF *$z_1(t)$ is M_{i1} AND …AND $z_p(t)$ is M_{ip}*
Then *$u_i(t)=-k_i F x(t)$*
$i=1, 2,\dots,r.$

where, $F \in R^{m \times n}$ is the local state feedback matrix, which is the same for all different local subsystems. $k_1 \sim k_i$ represent the proportional coefficients which differ with different control rules. We call this new controller Proportional PDC (PPDC). The spirit of our rule scheme is that rule consequent are proportional to one another. For r rules, the total number of unknown parameters is reduced to $r+m \times n$, compared with the previous $r \times m \times n$. As shown in Table 1, the effect of decreasing the number of parameters becomes bigger and bigger as the number of rules increases.

No. of Rules	PDC No. of Parameters	PPDC No. of Parameters
4	24	10
8	48	14
16	96	22
32	192	38

Table 1 Comparison of controller parameters

Here, the assumption of $m \times n=6$ is used, and the rule number r is 2^p, estimated by Assumption 1. As will be shown in the next section, the range of proportional coefficients will be given, i.e., the number of parameters needs to be designed can be reduced to $m \times n$ since we separate the proportional coefficients from the feedback gain. Furthermore, the plummeting of the parameters does not prevent the PPDC from providing perfect non-linear control which what other linear controllers and normal PDC can offer.

In our approach, each fuzzy control rule has a linear state feedback law in the THEN part. The overall fuzzy controller is represented by:

$$u(t)=\frac{-\sum_{i=1}^{r} w_i F_i x(t)}{\sum_{i=1}^{r} w_i}=-\sum_{i=1}^{r} h_i F_i x(t) \qquad (7)$$

Substituting controller (7) into the T-S fuzzy system given by (5) and (6), respectively, the overall closed-loop fuzzy system is obtained:

CFS:

$$\dot{x}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \{A_i - k_j B_i F\} x(t) \qquad (8)$$

DFS:

$$x(t+1)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j\{A_i-k_jB_iF\}x(t) \qquad (9)$$

4 Stability Analysis

The global asymptotic stability of a closed-loop T-S fuzzy system is discussed in this section. The theorems for stability conditions of fuzzy systems are in accordance with the definition of stability in the sense of Lyapunov.

Tanaka and other researchers have shown in [1] that the overall T-S system stability can be guaranteed by finding a common positive definite matrix P. Choosing $V(x(t))=x(t)^TPx(t)$ as the Lyapunov function of the T-S system, the following stability conditions for ensuring stability of (8) and (9) are derived by using the Lyapunov approach:

Theorem 1 <CFS>: The equilibrium of the closed-loop continuous fuzzy system described by using the proportional PDC is asymptotically stable in the large if there exists a common positive definite matrix P such that

$$(A_i-k_jB_iF)^TP+P(A_i-k_jB_iF)<0 \qquad (10)$$

for all *i*, *j*=1, 2,…,r except the pairs (*i*, *j*) such that $h_i*h_j=0$, ∀t.

Proof*:* It can be derived easily from the main theorems of [1]

Theorem 2 <DFS>: The equilibrium of the closed-loop discrete fuzzy system described by using the proportional PDC is asymptotically stable in the large if there exists a common positive definite matrix P such that

$$(A_i-k_jB_iF)^TP(A_i-k_jB_iF)-P<0 \qquad (11)$$

for all *i*, *j*=1, 2,…,r except the pairs (*i*, *j*) such that $h_i*h_j=0$, ∀t.

Proof*:* It can be derived easily from the main theorems of [1].

Remark 1: It is noted that (10) and (11) depend only on $A_i-k_jB_iF$. In other words, it does not depend on the rule weight, h_i.

Remark 2: In the Theorem 1 and 2, the number of inequalities is r^2, which can be reduced to $2r$ if we limit the range of k_i.

The following stability conditions hold:

Theorem 3 <CFS>: The equilibrium of the closed-loop continuous fuzzy system described by using the proportional PDC is asymptotically stable in the large if there exists a common positive definite matrix P such that

$$\begin{cases}(A_i-B_iF)^TP+P(A_i-B_iF)<0\\ (B_iF)^TP+P(B_iF)<0\\ 0<k_j<1\end{cases} \qquad (12)$$

for all *i*, *j*=1, 2,…,r except the pairs (*i*, *j*) such that $h_i*h_j=0$, ∀t.

Proof: The first inequality in (12) can be rewritten as:

$$A_i^TP+PA_i<(B_iF)^TP+P(B_iF)$$

Since the second and third inequalities in (12) hold, therefore, for all k_j:

$$(B_iF)^TP+P(B_iF)<k_j(B_iF)^TP+k_jP(B_iF)$$

Next, the following inequality holds for all *i*, *j*=1, 2,…,r.

$$A_i^TP+PA_i<k_j(B_iF)^TP+k_jP(B_iF)$$

$$(A_i-k_jB_iF)^TP+P(A_i-k_jB_iF)<0$$

From the above argument and Theorem 1, Theorem 3 is proven.

In order to prove Theorem 4, we need the following lemma.

Lemma 1 If P is a positive definite matrix, P>0, then A^TPA is also a positive definite matrix, $A^TPA>0$, where $A, P\in R^{n\times n}$.

Proof: Since P is a positive definite matrix, then $x^TPx>0$ for all $x\in R^n$ where $Ax\in R^n$,

$$x^T(A^TPA)x=(Ax)^TP(Ax)>0$$

so $A^TPA>0$. Therefore, the lemma holds.

Theorem 4 <DFS>: The equilibrium of the closed-loop discrete fuzzy system described by using the proportional PDC is asymptotically stable in the large if there exists a common positive definite matrix P such that

$$\begin{cases}(A_i + B_iF)^T P(A_i + B_iF) - P < 0 \\ (B_iF)^T PA_i + A_i^T P(B_iF) > 0 \\ 0 < k_j < 1\end{cases} \quad (13)$$

for all i, j=1, 2,…,r except the pairs (i, j) such that $h_i * h_j = 0$, ∀t.

Proof: Subtract the second inequality from the first inequality of (13), we have:

$$A_i^T PA_i + (B_iF)^T P(B_iF) < P$$

From Lemma 1 and (13)

$$k_j^2 (B_iF)^T P(B_iF) < (B_iF)^T P(B_iF)$$

so

$$A_i^T PA_i + k_j^2 (B_iF)^T P(B_iF) < P$$

Combining with (13), we have:

$$(A_i - k_jB_iF)^T P(A_i - k_jB_iF)$$

$$= A_i^T PA - k_j (B_iF)^T PA_i - k_j A_i^T P(B_iF)$$

$$+ k_j^2 (B_iF)^T P(B_iF)$$

$$< A_i^T PA + k_j^2 (B_iF)^T P(B_iF)$$

$$< P$$

From the above argument and Theorem 2, Theorem 4 holds.

Theorems 3 and 4 show that as long as the feedback matrix F satisfies the stability conditions, the proportional coefficients can be assigned freely in the range of (0,1).

5 LMI Solution

There are no analytic solutions for the proposed PPDC approach. However, the solution to the PPDC design can be solved numerically by the LMI approach that allows us to realise a total and systematic design satisfying the stability conditions. The basic idea is adapted from [5]. From Theorems 3 and 4, the design problem to determine the feedback gain F can be described as follows:

CFS:

Find Y>0 and M satisfying

$$\begin{cases}-YA_i^T - A_iY + M^TB_i^T + B_iM > 0 \\ M^TB_i^T + B_iM < 0\end{cases} \quad (14)$$

where $Y=P^{-1}$ and $M=FY$

DFS:

Find Y>0 and M satisfying

$$\begin{cases}\begin{bmatrix} Y & YA_i^T + M^TB_i^T \\ AY + B_iM & Y\end{bmatrix} > 0 \\ \begin{bmatrix} YA_i^T & M^TB_i^T \\ -A_iY & B_iM\end{bmatrix} > 0\end{cases} \quad (15)$$

where $Y=P^{-1}$ and $M=FY$.

The above conditions are convex feasibility problems in LMI. Numerically, these feasibility problems can be solved very efficiently by the recently developed interior-point method. Using LMI toolbox in Matlab, the feedback gain F and a common P can be obtained as:

$P = Y^{-1}$ and $F = MP$

6 An Illustrative Example

An illustrative example on stabilising a mass spring damper system using our approach is given in this section. It is assumed that the stiffness coefficient of the spring, the damping coefficient of the damper and the input term have nonlinearities. The system dynamic equation of the set-up is given by,

$$M\ddot{x}(t) + g(x(t), \dot{x}(t)) + f(x(t)) = f(\dot{x}(t))u(t) \quad (16)$$

where M is the mass, u is the force, $f(x(t))$ is the spring nonlinearity, $g(x(t), \dot{x}(t))$ is the damper nonlinearity, and $f(\dot{x}(t))$ is the input nonlinearity. Assume that:

$$g(x(t), \dot{x}(t)) = D(c_1x(t)) + c_2\dot{x}(t))$$

$$f(x(t)) = K(c_3x(t) + c_4x(t)^3)$$

$$f(\dot{x}(t)) = 1 + c_5\dot{x}(t)^3$$

Furthermore, assume that

$x \in [-1.5 \quad 1.5] \qquad \dot{x} \in [-1.5 \quad 1.5]$

The above parameters are given as follows: M=D=K=1, c_1=0, c_2=1, c_3=0.01, c_4=0.1, and c_5=0.13. The non-linear system then becomes,

$$\ddot{x}(t) = -\dot{x}(t) - 0.01x(t) - 0.1x(t)^3 + (1.4387 - 0.13\dot{x}(t)^2)u(t) \quad (17)$$

which is represented by a four-rules T-S fuzzy model as follows:

Rule 1: **IF** *x(t) is* M_{11} *AND* $\dot{x}(t)$ *is* M_{12}

Then $\dot{X}(t) = A_1X(t) + B_1u(t)$

Rule 2:**IF** *x(t) is M_{21} AND* $\dot{x}(t)$ *is M_{22}*

Then $\dot{X}(t) = A_2 X(t) + B_2 u(t)$

Rule 3:**IF** *x(t)* is M_{31} *AND* $\dot{x}(t)$ is M_{32}

Then $\dot{X}(t) = A_3 X(t) + B_3 u(t)$

Rule 4:**IF** *x(t) is M_{41} AND* $\dot{x}(t)$ *is M_{42}*

Then $\dot{X}(t) = A_4 X(t) + B_4 u(t)$

The membership functions of M_{ij}, system matrixes of A_i *and* B_i, *i=1,2,3,4, j=1,2,* are given by,

$$M_{11} = M_{21} = 1 - \frac{x(t)^2}{2.25}, M_{12} = M_{32} = 1 - \frac{x(t)^2}{6.75}$$

$$M_{31} = M_{41} = \frac{x(t)^2}{2.25}, M_{22} = M_{42} = \frac{x(t)^2}{6.75}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -1 \end{bmatrix}, \quad A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ -0.235 & -1 \end{bmatrix}$$

$$B_2 = B_4 = \begin{bmatrix} 0 \\ 1.4387 \end{bmatrix}, \quad B_1 = B_3 = \begin{bmatrix} 0 \\ 0.5613 \end{bmatrix}.$$

The fuzzy system is shown after fuzzification and defuzzification process taking A_i and B_i into account. It is noted that the original system (17) can be exactly represented by this T-S fuzzy system.

A four rules fuzzy controller is designed for this fuzzy system. It has the same input variables and fuzzy sets as the above T-S fuzzy model:

Rule 1: **IF** *x(t) is M_{11} AND* $\dot{x}(t)$ *is M_{12}*

Then $u(t) = -k_1 B_1 F X(t)$

Rule 2:**IF** *x(t) is M_{21} AND* $\dot{x}(t)$ *is M_{22}*

Then $u(t) = -k_2 B_2 F X(t)$

Rule 3:**IF** *x(t)* is M_{31} AND $\dot{x}(t)$ is M_{32}

Then $u(t) = -k_3 B_3 F X(t)$

Rule 4:**IF** *x(t) is M_{41} AND* $\dot{x}(t)$ *is M_{42}*

Then $u(t) = -k_4 B_4 F X(t)$

The proportional coefficients can be first assigned as:

$k_1=0.8, k_2=0.85, k_3=0.9, k_4=0.95$

According to (14), the problem of finding a common matrix P and feedback gain F can be achieved by solving the following linear matrix inequalities:

$$\begin{cases} -YA_i^T - A_i Y + M^T B_i^T + B_i M > 0 \\ M^T B_i^T + B_i M < 0 \end{cases}$$

i=1,2,3,4

Solving this convex feasibility problem, one of the solutions is:

$$F = \begin{bmatrix} 1.9707 & 3.3575 \end{bmatrix}, P = \begin{bmatrix} 0.1320 & 0.1813 \\ 0.1813 & 0.4202 \end{bmatrix}$$

By Theorem 3, we can conclude that the closed-loop system is asymptotically stable. The simulation results of responses under different initial conditions of $x_1(0) = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$ and $x_2(0) = \begin{bmatrix} -1.5 \\ 0 \end{bmatrix}$ are shown in Figures 1 and 2, respectively.

7 Conclusions

This paper presents a novel approach using PPDC. It greatly reduces the number of adjustable parameters in the PDC controller. To design such a PPDC, stability conditions based on Lyapunov functions are derived and solved using Linear Matrix Inequality. The proportional coefficients can be assigned in a specific range (0, 1). All these make our approach easy and simple for use. Finally, simulation results show that the new controller can stabilise a T-S fuzzy system effectively. Generally speaking, this paper has demonstrated a promising future of the PPDC design for practical application of fuzzy controllers.

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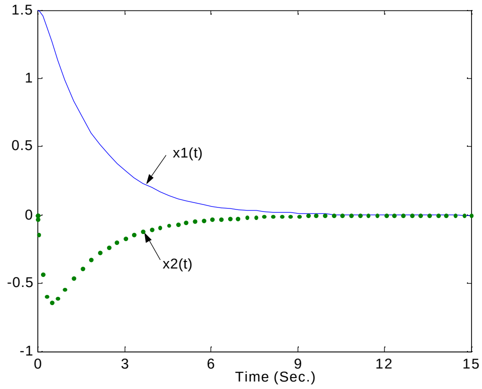

Figure.1 Simulation Responses ($x_1(0)=[1.5\ 0]^T$)

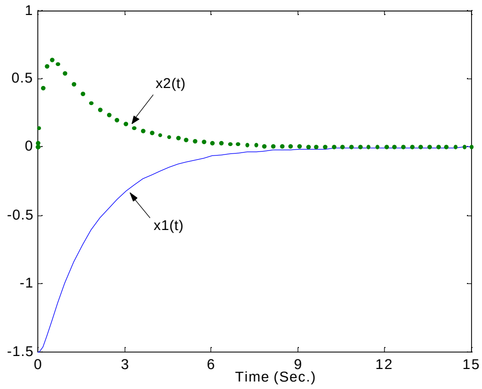

Figure.2 Simulation Responses ($x_2(0)=[-1.5\ 0]^T$)