A Novel Class of Blended Multiple Model System with Linear Local Models

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Abstract
It is shown that the dynamics of a conventional type of blended multiple model system are only weakly related to the local models from which it is formed. A novel class of velocity-based blended multiple model systems is proposed for which the dynamics are directly related to the local models. In contrast to conventional blended multiple model systems, the velocity-based blended multiple model systems employ *linear* local models, thereby providing a degree of continuity with established linear methods.

1. Introduction

Under quite general conditions, the solution to a nonlinear system approximates the solution to another nonlinear system over any finite time interval provided the terms in the differential equations describing the two systems are sufficiently similar (*e.g.* Khalil 1992 Theorem 2.5). Hence, by employing a sufficiently accurate representation of the terms in the differential equation describing a nonlinear system, a nonlinear system may be obtained which is an approximation to the original system. However, a common difficulty is the so-called "curse of dimensionality" whereby the number of parameters in the representation increases extremely rapidly as the order of the nonlinear dynamic system increases. This strongly motivates the investigation of representations which can accurately describe high order nonlinear systems with a relatively small number of parameters.

One promising class of representations is blended multiple model systems wherein a small number of relatively simple dynamic systems are, in some sense, blended together. Blended multiple model systems have been studied in a number of quite diverse fields (Johansen & Murray-Smith 1997). Typically, each simple system is a local model which describes the dynamics of the nonlinear system in some small region of the operating space. The role of the blending is to provide smooth interpolation, in some sense, between the local models with the aim of achieving an accurate representation with only a small number of local models. The blending is, therefore, central to the utility of the approach. This paper investigates the characteristics of the blended multiple model system in the operating regions in which no single local model dominates and several local models contribute significantly to the dynamics. Whilst central to the utility of blended multiple model representations, this issue is largely neglected in the literature.

2. Blended multiple model systems

Consider the conventional type of blended multiple model system

$$\dot{\tilde{\mathbf{x}}} = \sum_i \overline{\mathbf{F}}_i(\tilde{\mathbf{x}},\mathbf{r})\,\mu_i(\tilde{\rho}), \quad \tilde{\mathbf{y}} = \sum_i \overline{\mathbf{G}}_i(\tilde{\mathbf{x}},\mathbf{r})\,\mu_i(\tilde{\rho}) \qquad (1)$$

where $\tilde{\rho}(\tilde{\mathbf{x}},\mathbf{r})=\nabla_x\tilde{\rho}\,\tilde{\mathbf{x}}+\nabla_r\tilde{\rho}\,\mathbf{r}$, $\tilde{\mathbf{x}}\in\Re^n, \mathbf{r}\in\Re^m, \tilde{\mathbf{y}}\in\Re^p$ The $\overline{\mathbf{F}}_i$ and $\overline{\mathbf{G}}_i$ are differentiable nonlinear functions with Lipschitz continuous first derivatives. The μ_i are differentiable mappings, with Lipschitz continuous first derivatives, from the operating space, $\hat{\Phi}\subset\Re^n\times\Re^m$, onto the positive real line such that $\sum_i \mu_i(\tilde{\rho})=1$. The quantity, $\rho(\mathbf{x},\mathbf{r})$, embodying the dependence of the blending on the state and input, is assumed to be linear in the state, $\tilde{\mathbf{x}}$, and input, $\mathbf{r}$: no loss of generality is involved since any nonlinear dependence can be absorbed into the validity functions, μ_i. It is assumed, in addition, that the input and initial conditions are restricted such that the solution to (1) lies within the operating space, $\hat{\Phi}$.

The local models associated with the blended multiple model system are

$$\dot{\bar{\mathbf{x}}}_i = \overline{\mathbf{F}}_i(\bar{\mathbf{x}}_i,\mathbf{r}), \qquad \bar{\mathbf{y}}_i = \overline{\mathbf{G}}_i(\bar{\mathbf{x}}_i,\mathbf{r}) \qquad (2)$$

Typically (Johansen & Murray-smith 1997), affine local models are employed; that is,

$$\overline{\mathbf{F}}_i(\bar{\mathbf{x}},\mathbf{r}) = \bar{\alpha}_i + \overline{\mathbf{A}}_i\bar{\mathbf{x}} + \overline{\mathbf{B}}_i\mathbf{r}, \quad \overline{\mathbf{G}}_i(\bar{\mathbf{x}},\mathbf{r}) = \bar{\beta}_i + \overline{\mathbf{C}}_i\bar{\mathbf{x}} + \overline{\mathbf{D}}_i\mathbf{r}$$

where $\bar{\alpha}_i, \overline{\mathbf{A}}_i, \overline{\mathbf{B}}_i, \bar{\beta}_i, \overline{\mathbf{C}}_i, \overline{\mathbf{D}}_i$ are constants. To determine the dynamic behaviour of the blended multiple model system, (1), locally to an operating point, $(\tilde{\mathbf{x}}_1,\mathbf{r}_1)$, consider the first-order series expansion of (1) relative to the operating point,

$$\delta\dot{\tilde{\mathbf{x}}} = \sum_i \overline{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\rho}_1) + \sum_i \left[\nabla_x\overline{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\rho}_1) + \overline{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\rho}_1)\nabla_x\tilde{\rho}\right]\delta\tilde{\mathbf{x}}$$

$$\delta\tilde{\mathbf{y}} = \sum_i \left[\nabla_x\overline{\mathbf{G}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\rho}_1) + \overline{\mathbf{G}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\rho}_1)\nabla_x\tilde{\rho}\right]\delta\tilde{\mathbf{x}} + \sum_i \left[\nabla_r\overline{\mathbf{G}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\rho}_1) + \overline{\mathbf{G}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\rho}_1)\nabla_r\tilde{\rho}\right]\delta\mathbf{r} \qquad (3)$$

where $\tilde{\rho}_1=\nabla_x\tilde{\rho}\,\tilde{\mathbf{x}}_1+\nabla_r\tilde{\rho}\,\mathbf{r}_1$, $\delta\mathbf{r}=\mathbf{r}-\mathbf{r}_1$,

$$\tilde{\mathbf{y}} = \sum_i \overline{\mathbf{G}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\,\mu_i(\tilde{\rho}_1) + \delta\tilde{\mathbf{y}}, \quad \tilde{\mathbf{x}} = \delta\tilde{\mathbf{x}} + \tilde{\mathbf{x}}_1 \qquad (4)$$

It should be noted that the expansion is carried out relative to the fixed operating point, $(\tilde{\mathbf{x}}_1,\mathbf{r}_1)$, as opposed to a trajectory passing through $(\tilde{\mathbf{x}}_1,\mathbf{r}_1)$, and the derivative of $\tilde{\mathbf{x}}_1$ is, therefore, zero. It is emphasised that (3)-(4) is a well-defined system in its own right, distinct from the nonlinear system (1), with state $\delta\tilde{\mathbf{x}}$.

Furthermore, the point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, need not be an equilibrium operating point and, indeed, may lie far from the locus of equilibrium operating points. When $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$ is an equilibrium operating point, the inhomogeneous term in (3) is zero in which case the dynamics of (3)-(4) are determined solely by the terms in $\delta\breve{\mathbf{x}}$ and $\delta\mathbf{r}$ (note that, in general, the contributions to these terms from both μ_i and $\nabla\mu_i$ are significant). However, when $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$ is a non-equilibrium operating point the inhomogeneous term may be extremely large and can dominate the solution to (3)-(4).

Combining (3) with the local input, output and state transformations, (4), the first-order representation, (3)-(4), may be reformulated as

$$\begin{aligned}\dot{\breve{\mathbf{x}}} = &\sum_i \bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1) - \Big[\nabla_{\mathbf{x}}\bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1) + \bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\boldsymbol{\rho}}_1)\nabla_{\mathbf{x}}\tilde{\boldsymbol{\rho}}\Big]\tilde{\mathbf{x}}_1 \\ &- \sum_i \Big[\nabla_{\mathbf{r}}\bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1) + \bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\boldsymbol{\rho}}_1)\nabla_{\mathbf{r}}\tilde{\boldsymbol{\rho}}\Big]\mathbf{r}_1 \\ &+ \sum_i \Big[\nabla_{\mathbf{x}}\bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1) + \bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\boldsymbol{\rho}}_1)\nabla_{\mathbf{x}}\tilde{\boldsymbol{\rho}}\Big]\breve{\mathbf{x}} \\ &+ \sum_i \Big[\nabla_{\mathbf{r}}\bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1) + \bar{\mathbf{F}}_i(\tilde{\mathbf{x}}_1,\mathbf{r}_1)\nabla\mu_i(\tilde{\boldsymbol{\rho}}_1)\nabla_{\mathbf{r}}\tilde{\boldsymbol{\rho}}\Big]\mathbf{r}\end{aligned} \tag{5}$$

and similarly for the output equation. In contrast to (3)-(4), the state, input and output are now the same at every operating point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$.

The blending in a blended multiple model system is required to provide smooth interpolation, in some sense, between the local models to achieve an accurate representation over the whole operating space with only a small number of local models. To facilitate analysis and design, the dynamics of the blended multiple model system in these regions are required to be directly related to the local models. Locally to the operating point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, the solution, $\tilde{\mathbf{x}}$, to the blended multiple model system is approximated by the solution, $\breve{\mathbf{x}}$, to the first-order series expansion (5) (Leith & Leithead 1998b, see also section 3.1). However, it is evident from (5) that the relationship between the first-order series expansions and the local models is not straightforward. The dynamics of the blended multiple model system are *not* described by a simple blend of dynamics of the local models. Indeed, in operating regions where the derivative, $\nabla\mu_i$, of any validity function is large, the dynamics of the first-order series expansions may be strongly influenced by the varying nature of the validity functions and only weakly related to the dynamics of the local models. Hence, the utility of this class of blended multiple model systems is diminished considerably. Furthermore, although widely employed in the literature, it is noted that affine local models do not provide continuity with established linear methods and, in particular, do not possess the superposition property which is fundamental to linear systems. It is emphasised that the magnitude of the inhomogeneous term in an affine local model may be extremely large and can strongly influence the solution. When the inhomogeneous term varies from one local model to another, it introduces an implicit feedback from the state and/or input of the overall blended multiple model system. Hence, the inhomogeneous term cannot, in general, simply be treated as a disturbance or small approximation error.

3. Velocity-based blended multiple model systems

Two difficulties with the conventional approach to blended multiple model systems using affine local models are noted above; namely, the lack of a direct relationship, between the dynamics of the blended multiple model system and the dynamics of the local models, and the lack of linearity of the local models themselves. To circumvent these difficulties the velocity-based representation of nonlinear dynamics systems of Leith & Leithead (1998b) is adopted; that is, the dynamics are represented by a velocity-based nonlinear system or, equivalently, a velocity-based linearisation family.

3.1 Velocity-based linearisation families

Before proceeding, the velocity-based approach of Leith & Leithead (1998b) is briefly summarised. Consider nonlinear plants with dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{6}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable nonlinear functions with Lipschitz continuous first derivatives and $\mathbf{r} \in \Re^m$ denotes the input to the plant, $\mathbf{y} \in \Re^p$ the output and $\mathbf{x} \in \Re^n$ the states. The set of equilibrium operating points of the nonlinear plant, (6), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which $\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0$; the corresponding equilibrium output is $\mathbf{y_o} = \mathbf{G}(\mathbf{x_o}, \mathbf{r_o})$. Let $\Phi: \Re^n \times \Re^m$ denote the space consisting of the union of the states, $\mathbf{x}$, with the inputs, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear plant, (6), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the plant to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

The nonlinear system, (6), may be reformulated, equivalently, as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\boldsymbol{\rho}), \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\boldsymbol{\rho}) \tag{7}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ nonlinear functions and $\boldsymbol{\rho}(\mathbf{x},\mathbf{r}) \in \Re^q$, $q \le m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ functions of $\boldsymbol{\rho}$ alone. Trivially, this reformulation can always be achieved by letting $\boldsymbol{\rho} = [\mathbf{x}^T \ \mathbf{r}^T]^T$, in which case $q=m+n$. However, the nonlinearity of the system is frequently dependent on only a subset of the states and inputs, in which case the dimension, q, of $\boldsymbol{\rho}$ is less than m+n. Since $\nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ are functions of $\boldsymbol{\rho}$ alone, the variable, $\boldsymbol{\rho}(\mathbf{x},\mathbf{r})$, equals the constant value, $\boldsymbol{\rho}_1$, upon a surface of co-dimension q in Φ and $\nabla_{\mathbf{x}}\boldsymbol{\rho}$ and $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ are constant over each surface. Hence, the normal to each surface is identical at every point on the surface and each surface is, therefore, affine. Moreover, to ensure that $\boldsymbol{\rho}$ is a unique function of $\mathbf{x}$ and $\mathbf{r}$, these surfaces must be parallel for all $\boldsymbol{\rho}$. Consequently, it may be assumed, without loss of generality, that $\nabla_{\mathbf{x}}\boldsymbol{\rho}$ and $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ are constant.

Suppose that the nonlinear system, (7), is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, in Φ and at time, t_1, the trajectory has reached the point, $(\mathbf{x_1}, \mathbf{r_1})$. It is

emphasised that the point, $(\mathbf{x}_1, \mathbf{r}_1)$, need not be an equilibrium operating point and, indeed, may lie far from the locus of equilibrium operating points. From Taylor series expansion theory, the subsequent behaviour of the nonlinear system can be approximated, locally to the point, $(\mathbf{x}_1, \mathbf{r}_1)$, by the first order representation

$$\delta\dot{\breve{\mathbf{x}}} = (\mathbf{A}\mathbf{x}_1+\mathbf{B}\mathbf{r}_1+\mathbf{f}(\boldsymbol{\rho}_1)) + (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{x}}\boldsymbol{\rho})\delta\breve{\mathbf{x}} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{r}}\boldsymbol{\rho})\delta\mathbf{r} \quad (8)$$

together with a similar output equation, where

$$\boldsymbol{\rho}_1=\boldsymbol{\rho}(\mathbf{x}_1,\mathbf{r}_1),\quad \delta\mathbf{r} = \mathbf{r} - \mathbf{r}_1,\quad \breve{\mathbf{x}} = \delta\breve{\mathbf{x}} + \mathbf{x}_1,\ \dot{\breve{\mathbf{x}}} = \delta\dot{\breve{\mathbf{x}}} \quad (9)$$

provided $\mathbf{x}_1+\delta\breve{\mathbf{x}} \subseteq \mathrm{N}_{\mathbf{x}}$ $\mathbf{r}_1+\delta\mathbf{r}\subseteq \mathrm{N}_{\mathbf{r}}$, where the neighbourhoods, $\mathrm{N}_{\mathbf{x}}$ and $\mathrm{N}_{\mathbf{r}}$, of, respectively, $\mathbf{x}_1$ and $\mathbf{r}_1$ are sufficiently small. When $\delta\breve{\mathbf{x}}(t_1)$ is zero, the solution, $\breve{\mathbf{x}}(t)$, to (8)-(9), initially at time t_1, is tangential to the solution, $\mathbf{x}(t)$, of (7). Indeed, locally to time t_1, $\breve{\mathbf{x}}(t)$ provides a first-order approximation to $\dot{\mathbf{x}}(t)$ and a second-order approximation to $\mathbf{x}(t)$.

The solution, $\breve{\mathbf{x}}(t)$, to the first-order series expansion, (8)-(9), provides a valid approximation only while the solution, $\mathbf{x}(t)$, to the nonlinear system remains in the vicinity of the operating point, $(\mathbf{x}_1, \mathbf{r}_1)$. However, the solution, $\mathbf{x}(t)$, to the nonlinear system need not stay in the vicinity of a single operating point. Consider the time interval, $[0,T]$; the initial time can, without loss of generality, always be taken as zero. An approximation to $\mathbf{x}(t)$ is obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is the solution to the first-order series expansion relative to the operating point reached at the initial time for the sub-interval (with the initial conditions for each sub-interval chosen to ensure continuity of the approximate solution). The number of local solutions employed is dependent on the duration of the sub-intervals, but the local solutions are accurate to second order; that is, the approximation error is proportional to the duration of the sub-interval cubed. Hence, as the number of sub-intervals increases, the approximation error associated with each rapidly decreases and the overall approximation error also decreases. Indeed, the overall approximation error tends to zero as the maximum size of the sub-intervals tends to zero. Hence, the family of first-order series expansions, with members defined by (8)-(9), can provide an arbitrarily accurate approximation to the solution of the nonlinear system. Moreover, this approximation property holds throughout Φ and is not confined to the vicinity of a single equilibrium operating point or even of the locus of equilibrium operating points.

It should be noted that the state, input and output transformations, (9), depend on the operating point relative to which the series expansion is carried out. When the solution to the nonlinear system is confined to a neighbourhood about a single operating point, the transformations, (9), are static and the dynamic behaviour is described by the system, (8), alone. However, when the solution to the nonlinear system traces a trajectory which is not confined to a neighbourhood about a single operating point, the transformations are no longer static and the dynamic behaviour is no longer described solely by (8). Instead, the dynamic behaviour is described by (8)-(9). Combining (8) with the local input, output and state transformations, (9), each member, (8)-(9), of the family of first-order representations may be reformulated as,

$$\dot{\breve{\mathbf{x}}} = \{\mathbf{f}(\boldsymbol{\rho}_1)-\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{x}}\boldsymbol{\rho}\,\mathbf{x}_1-\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{r}}\boldsymbol{\rho}\,\mathbf{r}_1\} + (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{x}}\boldsymbol{\rho})\breve{\mathbf{x}} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{r}}\boldsymbol{\rho})\mathbf{r} \quad (10)$$

together with a similar output equation. In contrast to (8), the state, input and output are now the same for all members of the reformulated family. The dynamics, (10), of an individual member of the family are affine rather than linear even when $(\mathbf{x}_1, \mathbf{r}_1)$ is an equilibrium operating point. The inhomogeneous terms in (10) may, in general, be extremely large and can dominate the solution.

On differentiating (10) and the associated output equation,

$$\ddot{\breve{\mathbf{x}}} = \dot{\breve{\mathbf{w}}} \quad (11)$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{x}}\boldsymbol{\rho})\breve{\mathbf{w}} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho}_1)\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \quad (12)$$

$$\dot{\breve{\mathbf{y}}} = (\mathbf{C}+\nabla\mathbf{g}(\boldsymbol{\rho}_1)\nabla_{\mathbf{x}}\boldsymbol{\rho})\breve{\mathbf{w}} + (\mathbf{D}+\nabla\mathbf{g}(\boldsymbol{\rho}_1)\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \quad (13)$$

The system, (11)-(13), is dynamically equivalent to the system, (10), in the sense that with appropriate initial conditions, the solution, $\breve{\mathbf{x}}$, to (11)-(13), is the same as the solution, $\breve{\mathbf{x}}$, to (10). However, in contrast to (10), the transformed system, (11)-(13), is linear. The relationship between the nonlinear system and its velocity-based linearisation, (11)-(13), is direct. Differentiating (7), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \quad (14)$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \quad (15)$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \quad (16)$$

Dynamically, (14)-(16), with appropriate initial conditions, and (7) are equivalent (have the same solution, $\mathbf{x}$). When $\breve{\mathbf{x}}(t_1) = \mathbf{x}(t_1)$, $\breve{\mathbf{w}}(t_1) = \mathbf{w}(t_1)$, $\breve{\mathbf{y}}(t_1)= \mathbf{y}(t_1)$ it follows from (11)-(13) and (14)-(16) that $\breve{\mathbf{x}}(t)$ and $\breve{\mathbf{y}}(t)$ provide a second- and first-order approximation to, respectively, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ and $\breve{\mathbf{w}}(t)$ provides a first-order approximation to $\mathbf{w}(t)$. Clearly, the velocity-based linearisation, (11)-(13), is simply the frozen form of (14)-(16) at the operating point, $(\mathbf{x}_1, \mathbf{r}_1)$.

There exists a velocity-based linearisation, (11)-(13), for every point in Φ. Hence, a velocity-based linearisation family, with members defined by (11)-(13), can be associated with the nonlinear system, (7). Similarly to the family of first-order expansions, the solutions to the members of the family of velocity-based linearisations, (11)-(13), can be pieced together (with the initial conditions for each sub-interval chosen to ensure continuity of $\breve{\mathbf{x}}$, $\breve{\mathbf{w}}$ and $\breve{\mathbf{y}}$) to approximate the solution to the nonlinear system, (14)-(16) to an arbitrary degree of accuracy.

There exists a rigorous, and direct, relationship between the dynamic characteristics of a nonlinear system and those of a related family of linear systems, namely, the velocity-based linearisation family, and a

related family of affine systems, namely the first-order series expansion family. Since the solutions to the members of the families can be combined to approximate the solution to the nonlinear system arbitrarily accurately, the families embody the entire dynamics of the nonlinear system, (7), with no loss of information and therefore provide alternative representations of the nonlinear system. Whilst these representations are equivalent in the sense that they each embody the entire dynamics of the nonlinear system, they are not necessarily equivalent with respect to other considerations. In particular, the *direct* relationship between the velocity-form of the nonlinear system and the velocity-based linearisation family and the linearity of the members of the latter family provides continuity with established linear theory which, for example, facilitates analysis (Leith & Leithead 1998b) and controller design (Leith & Leithead 1998c).

3.2 Velocity-based blended multiple model systems

Instead of using the direct forms of the local models, consider constructing a blended multiple model system using the velocity-based forms; that is, consider the velocity-based blended multiple model system

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{w}}\\ \dot{\tilde{\mathbf{w}}} &= \Big(\sum_i \boldsymbol{\phi}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}})\Big)\tilde{\mathbf{w}} + \Big(\sum_i \boldsymbol{\phi}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}})\Big)\dot{\mathbf{r}}\\ \dot{\tilde{\mathbf{y}}} &= \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}})\Big)\tilde{\mathbf{w}} + \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}})\Big)\dot{\mathbf{r}}\end{aligned} \qquad (17)$$

where $\boldsymbol{\phi}_{\mathbf{x}}^i, \boldsymbol{\phi}_{\mathbf{r}}^i, \boldsymbol{\gamma}_{\mathbf{x}}^i$ and $\boldsymbol{\gamma}_{\mathbf{r}}^i$ are constant matrices. The corresponding local models, from which the blended multiple model system is constituted, are

$$\dot{\bar{\mathbf{x}}}_i = \bar{\mathbf{w}}_i, \quad \dot{\bar{\mathbf{w}}}_i = \boldsymbol{\phi}_{\mathbf{x}}^i \bar{\mathbf{w}}_i + \boldsymbol{\phi}_{\mathbf{r}}^i \dot{\mathbf{r}}, \quad \dot{\bar{\mathbf{y}}}_i = \boldsymbol{\gamma}_{\mathbf{x}}^i \bar{\mathbf{w}}_i + \boldsymbol{\gamma}_{\mathbf{r}}^i \dot{\mathbf{r}} \qquad (18)$$

It should be noted that the local models are *linear*. Assume that the local models satisfy

$$\boldsymbol{\phi}_{\mathbf{x}}^i = \boldsymbol{\phi}_{\mathbf{x}}(\mathbf{x}_i, \mathbf{r}_i), \quad \boldsymbol{\phi}_{\mathbf{r}}^i = \boldsymbol{\phi}_{\mathbf{r}}(\mathbf{x}_i, \mathbf{r}_i), \quad \boldsymbol{\gamma}_{\mathbf{x}}^i = \boldsymbol{\gamma}_{\mathbf{x}}(\mathbf{x}_i, \mathbf{r}_i), \quad \boldsymbol{\gamma}_{\mathbf{r}}^i = \boldsymbol{\gamma}_{\mathbf{r}}(\mathbf{x}_i, \mathbf{r}_i)$$

where

$$\boldsymbol{\phi}_{\mathbf{x}}(\mathbf{x}_i, \mathbf{r}_i) = \nabla_{\mathbf{x}} \boldsymbol{\phi}(\mathbf{x}, \mathbf{r})\big|_{(\mathbf{x}_i, \mathbf{r}_i)}, \quad \boldsymbol{\phi}_{\mathbf{r}}(\mathbf{x}_i, \mathbf{r}_i) = \nabla_{\mathbf{r}} \boldsymbol{\phi}(\mathbf{x}, \mathbf{r})\big|_{(\mathbf{x}_i, \mathbf{r}_i)}$$

and $\boldsymbol{\phi}_{\mathbf{x}}(\bullet,\bullet) = \nabla_{\mathbf{x}} \boldsymbol{\gamma}(\mathbf{x}, \mathbf{r})\big|_{(\mathbf{x}_i, \mathbf{r}_i)}$ are $\boldsymbol{\gamma}_{\mathbf{r}}(\bullet,\bullet) = \nabla_{\mathbf{r}} \boldsymbol{\gamma}(\mathbf{x}, \mathbf{r})\big|_{(\mathbf{x}_i, \mathbf{r}_i)}$ differentiable non-linear functions with Lipschitz continuous first derivatives. Each local model then corresponds to the velocity-based linearisation, at the operating point $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$, of the nonlinear system

$$\dot{\mathbf{x}} = \boldsymbol{\phi}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \boldsymbol{\gamma}(\mathbf{x}, \mathbf{r}) \qquad (19)$$

Since the functions, $\boldsymbol{\phi}(\bullet,\bullet)$ and $\boldsymbol{\gamma}(\bullet,\bullet)$, are arbitrary (other than being required to be smooth), this requirement is simply a smoothness condition which serves a role similar to the regularisation measures required with blended multiple model systems, (1). In the situation considered in this paper, where the number of local models is finite and small, this condition is very weak and only requires that the coefficients of the local models can be interpolated by integrable functions.

Suppose that the blended multiple model system is evolving along a trajectory in Φ and at time, t_1, the trajectory has reached the point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$. Similarly to the discussion in section 3.1 concerning the relationship of (11)-(13) to (14)-(16), the solution to the frozen form of (17) at the operating point $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$

$$\begin{aligned}\dot{\breve{\mathbf{x}}} &= \breve{\mathbf{w}}\\ \dot{\breve{\mathbf{w}}} &= \Big(\sum_i \boldsymbol{\phi}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\breve{\mathbf{w}} + \Big(\sum_i \boldsymbol{\phi}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\dot{\mathbf{r}}\\ \dot{\breve{\mathbf{y}}} &= \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\breve{\mathbf{w}} + \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\dot{\mathbf{r}}\end{aligned} \qquad (20)$$

with $\tilde{\boldsymbol{\rho}}_1 = \nabla_{\mathbf{x}} \tilde{\boldsymbol{\rho}}\ \tilde{\mathbf{x}}_1 + \nabla_{\mathbf{r}} \tilde{\boldsymbol{\rho}}\ \mathbf{r}_1$ and initial conditions

$$\breve{\mathbf{x}}(t_1) = \tilde{\mathbf{x}}(t_1), \quad \breve{\mathbf{w}}(t_1) = \tilde{\mathbf{w}}(t_1), \quad \breve{\mathbf{y}}(t_1) = \tilde{\mathbf{y}}(t_1)$$

is, initially at time t_1, tangential to the solution to the blended multiple model system, (17). Indeed, $\breve{\mathbf{x}}$ is a second-order approximation to $\tilde{\mathbf{x}}$, $\breve{\mathbf{w}}$ is a first-order approximation to $\tilde{\mathbf{w}}$ and $\breve{\mathbf{y}}$ is a first-order approximation to $\tilde{\mathbf{y}}$. Hence, the dynamics of the blended multiple model system, locally to the operating point $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, are indicated by the dynamics of the corresponding frozen-form linear system, (20). Clearly, the frozen-form, (20), of the blended multiple model system is directly related to the local models, (18), from which it is constructed. Specifically, the frozen-form at the operating point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, is a weighted linear combination of the local models with the weighting assigned to each local model just the value of the corresponding validity function at $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$. Hence, the dynamics of the blended multiple model system are directly related to the local models as required; that is, the dynamics are described by a straightforward blend of the local models.

The weighted linear combination of the *solutions* to the local models, (18), satisfies

$$\begin{aligned}\dot{\breve{\mathbf{x}}}^* &= \breve{\mathbf{w}}^*\\ \dot{\breve{\mathbf{w}}}^* &= \Big(\sum_i \boldsymbol{\phi}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\breve{\mathbf{w}}^* + \Big(\sum_i \boldsymbol{\phi}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\dot{\mathbf{r}} + \boldsymbol{\varepsilon}_{\mathbf{w}^*}\\ \dot{\breve{\mathbf{y}}}^* &= \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{x}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\breve{\mathbf{w}}^* + \Big(\sum_i \boldsymbol{\gamma}_{\mathbf{r}}^i \mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)\dot{\mathbf{r}} + \boldsymbol{\varepsilon}_{\mathbf{y}^*}\end{aligned} \qquad (21)$$

where $\breve{\mathbf{x}}^* = \sum_i \bar{\mathbf{x}}_i \mu_i(\tilde{\boldsymbol{\rho}}_1)$, $\breve{\mathbf{w}}^* = \sum_i \bar{\mathbf{w}}_i \mu_i(\tilde{\boldsymbol{\rho}}_1)$,

$\breve{\mathbf{y}}^* = \sum_i \bar{\mathbf{y}}_i \mu_i(\tilde{\boldsymbol{\rho}}_1)$, $\boldsymbol{\varepsilon}_{\mathbf{w}^*} = \sum_i \sum_j \big(\boldsymbol{\phi}_{\mathbf{x}}^i - \boldsymbol{\phi}_{\mathbf{x}}^j\big)\bar{\mathbf{w}}_i \mu_j(\tilde{\boldsymbol{\rho}}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1)$,

$\boldsymbol{\varepsilon}_{\mathbf{y}^*} = \sum_i \sum_j \big(\boldsymbol{\gamma}_{\mathbf{x}}^i - \boldsymbol{\gamma}_{\mathbf{x}}^j\big)\bar{\mathbf{w}}_i \mu_i(\tilde{\boldsymbol{\rho}}_1)\mu_j(\tilde{\boldsymbol{\rho}}_1)$

It follows that, provided $\boldsymbol{\varepsilon}_{\mathbf{w}^*}$ and $\boldsymbol{\varepsilon}_{\mathbf{y}^*}$ are sufficiently small, the solution to the frozen form is described by the weighted linear combination of the solutions to the local models. Since $\boldsymbol{\phi}_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{r})$, $\boldsymbol{\gamma}_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{r})$ are Lipschitz continuous in $\hat{\Phi}$, there exists a positive finite constant, L, such that

$$\big|\boldsymbol{\varepsilon}_{\mathbf{w}^*}\big| \le \Big(\sum_i \sum_j L\big(|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j| + |\mathbf{r}_i - \mathbf{r}_j|\big)|\bar{\mathbf{w}}_i|\mu_j(\tilde{\boldsymbol{\rho}}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)$$

$$\big|\boldsymbol{\varepsilon}_{\mathbf{y}^*}\big| \le \Big(\sum_i \sum_j L\big(|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j| + |\mathbf{r}_i - \mathbf{r}_j|\big)|\bar{\mathbf{w}}_i|\mu_j(\tilde{\boldsymbol{\rho}}_1)\mu_i(\tilde{\boldsymbol{\rho}}_1)\Big)$$

Hence, the magnitudes of $\boldsymbol{\varepsilon}_{\mathbf{w}^*}$ and $\boldsymbol{\varepsilon}_{\mathbf{y}^*}$ are proportional to the distance between the centres, $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$, of the local

models and the overlap in the validity functions, μ_i. Evidently, $\boldsymbol{\varepsilon}_{w^*}$ and $\boldsymbol{\varepsilon}_{y^*}$ tend to zero as the distance between the centres tends to zero provided the overlap is confined to neighbouring local models and the solutions, $\overline{\mathbf{w}}_i$, are bounded. When the magnitudes of $\boldsymbol{\varepsilon}_{w^*}$ and $\boldsymbol{\varepsilon}_{y^*}$ are sufficiently small, they can be neglected in (21). In these circumstances, the solution to (20) (and therefore the solution, locally to $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, to the blended multiple model system) can be approximated by the weighted linear combination of the solutions to the local models. Of course, the degree of accuracy to which this approximation corresponds, like the degree of accuracy of representation of the blended multiple model system, is related to the number of local models and a suitable, contextually dependent, compromise must be reached to keep this number small.

Furthermore, consider the nonlinear system, (7), for which the velocity-based linearisation family is (11)-(13) and let $\nabla_x\tilde{\rho} = \nabla_x\rho, \quad \nabla_r\tilde{\rho} = \nabla_r\rho$

$$\boldsymbol{\phi}_x^i = \mathbf{A} + \nabla_x\mathbf{f}(\rho(\tilde{\mathbf{x}}_i, \mathbf{r}_i))\nabla_x\rho, \quad \boldsymbol{\phi}_r^i = \mathbf{B} + \nabla_r\mathbf{f}(\rho(\tilde{\mathbf{x}}_i, \mathbf{r}_i))\nabla_r\rho$$

$$\boldsymbol{\gamma}_x^i = \mathbf{C} + \nabla_x\mathbf{g}(\rho(\tilde{\mathbf{x}}_i, \mathbf{r}_i))\nabla_x\rho, \quad \boldsymbol{\gamma}_r^i = \mathbf{D} + \nabla_r\mathbf{g}(\rho(\tilde{\mathbf{x}}_i, \mathbf{r}_i))\nabla_r\rho$$

When $\mathbf{r}_i = \mathbf{r}_1$ the frozen-form of the blended multiple model system at the operating point $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$ is precisely the velocity-based linearisation of (7) at $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$. At other operating points, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, the frozen form, (20), is the approximation to the velocity-based linearisation of (7) obtained by interpolating, using the validity functions, between the velocity-based linearisations at the operating points, $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$. A frozen form, (20), is associated with every operating point and the resulting family of frozen forms is an approximation to the velocity-based linearisation family of (7). Hence, the blended multiple model system, (17), is an approximation to the velocity-form, (14)-(16), of the nonlinear system, (7), and the solution $\tilde{\mathbf{x}}$ to the blended multiple model system approximates the solution $\mathbf{x}$ to (7) over any finite time interval (see, for example, Khalil 1992 Theorem 2.5). Indeed, locally to the operating point, $(\tilde{\mathbf{x}}_1, \mathbf{r}_1)$, the solution $\mathbf{x}$ to the nonlinear system, (7), is approximated by the solution to the velocity-based linearisation, (11)-(13), which in turn is approximated by the solution $\bar{\mathbf{x}}$ to the approximate velocity-based linearisation, (20).

3.3 Velocity-based blended multiple controllers

The requirement for a direct relationship between the dynamics of a blended multiple model system and the local models is motivated, in part, by control design requirements. For example, in the context of local model networks, Hunt & Johansen (1997) consider the design of a nonlinear controller whereby a linear controller is designed for each local model and the members of the family of linear controllers obtained are blended to obtain a nonlinear controller. However, the design procedure is complicated considerably by the lack of a direct relationship between the dynamics of such blended multiple model systems and the local models on which the control design is based. Moreover, the local models employed are affine and do not provide continuity with established linear methods. In particular, they do not possess the superposition property which is fundamental to linear systems. The magnitude of the inhomogeneous term in an affine local model may be extremely large and can strongly influence the solution. Furthermore, when the inhomogeneous term varies from one local model to another, it introduces an implicit feedback from the state and/or input of the overall blended multiple model system. Hence, the inhomogeneous term cannot, in general, simply be treated as a small disturbance to be rejected by the controller and this leads to Hunt & Johansen (1997) introducing slow variation requirements which are quite distinct from those encountered in conventional gain-scheduling.

In contrast to conventional blended multiple model systems, the dynamics of the velocity-based blended multiple model system, (17), in the blended regions are *directly* related to the local models. Moreover, the local models are linear. Control design based on a blended multiple model plant representation is thereby considerably facilitated. Consider the nonlinear plant, (17), and the nonlinear controller

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}}_c &= \tilde{\mathbf{w}}_c \\
\dot{\tilde{\mathbf{w}}}_c &= (\sum_i \boldsymbol{\phi}_{x_c}^i \mu_i(\tilde{\boldsymbol{\rho}}))\tilde{\mathbf{w}}_c + (\sum_i \boldsymbol{\phi}_{r_c}^i \mu_i(\tilde{\boldsymbol{\rho}}))\dot{\mathbf{r}}_c \qquad (22) \\
\tilde{\mathbf{y}}_c &= (\sum_i \boldsymbol{\gamma}_{x_c}^i \mu_i(\tilde{\boldsymbol{\rho}}))\tilde{\mathbf{w}}_c + (\sum_i \boldsymbol{\gamma}_{r_c}^i \mu_i(\tilde{\boldsymbol{\rho}}))\dot{\mathbf{r}}_c
\end{aligned}$$

where $\tilde{\mathbf{r}}_c \in \Re^{m_c}$ denotes the input to the controller, $\tilde{\mathbf{y}}_c \in \Re^{p_c}$ the output and $\tilde{\mathbf{x}}_c \in \Re^{n_c}$ the state. Since the requirement is to design a feedback controller, to facilitate the analysis it is assumed the input, $\tilde{\mathbf{r}}$, to the plant is the output, $\tilde{\mathbf{y}}_c$, of the controller and the input vector, $\tilde{\mathbf{r}}_c$, to the controller is the error, $\tilde{\mathbf{y}}_{\text{ref}} - \tilde{\mathbf{y}}$. Assume, in addition, that $\boldsymbol{\gamma}_r^i$ and $\boldsymbol{\gamma}_{r_c}^i$ are zero. These assumptions are employed only to simplify the analysis and it is straightforward to extend to analysis to include more general configurations, including two degree of freedom controllers and situations where $\boldsymbol{\gamma}_r^i$ and $\boldsymbol{\gamma}_{r_c}^i$ are not zero. The combined closed-loop dynamics are

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{w}}, \quad \dot{\tilde{\mathbf{x}}}_c = \tilde{\mathbf{w}}_c$$

$$\begin{bmatrix} \dot{\tilde{\mathbf{w}}} \\ \dot{\tilde{\mathbf{w}}}_c \end{bmatrix} = \begin{bmatrix} \sum_i \boldsymbol{\phi}_x^i \mu_i(\tilde{\boldsymbol{\rho}}) & \sum_i \boldsymbol{\phi}_{r_c}^i \mu_i(\tilde{\boldsymbol{\rho}}) \\ \sum_i \boldsymbol{\phi}_{x_c}^i \mu_i(\tilde{\boldsymbol{\rho}}) & \sum_i \boldsymbol{\phi}_{r_c}^i \mu_i(\tilde{\boldsymbol{\rho}}) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}} \\ \tilde{\mathbf{w}}_c \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_i \boldsymbol{\phi}_{r_c}^i \mu_i(\tilde{\boldsymbol{\rho}}) \end{bmatrix} \dot{\mathbf{y}}_{\text{ref}} + \boldsymbol{\varepsilon}_w$$

$$\tilde{\mathbf{y}} = \begin{bmatrix} \sum_i \boldsymbol{\gamma}_x^i \mu_i(\tilde{\boldsymbol{\rho}}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{x}}_c \end{bmatrix}$$

where $\boldsymbol{\varepsilon}_w = \sum_i \sum_j \begin{bmatrix} \boldsymbol{\phi}_x^i(\boldsymbol{\gamma}_{x_c}^j - \boldsymbol{\gamma}_{x_c}^i)\tilde{\mathbf{w}}_c \\ \boldsymbol{\phi}_{r_c}^i(\boldsymbol{\gamma}_x^j - \boldsymbol{\gamma}_x^i)\tilde{\mathbf{w}} \end{bmatrix} \mu_i(\tilde{\boldsymbol{\rho}})\mu_j(\tilde{\boldsymbol{\rho}})$

The perturbation term, $\boldsymbol{\varepsilon}_w$, vanishes when $\sum_i \boldsymbol{\gamma}_x^i \mu_i(\tilde{\boldsymbol{\rho}}) = \mathbf{C}$, $\sum_i \boldsymbol{\gamma}_{x_c}^i \mu_i(\tilde{\boldsymbol{\rho}}) = \mathbf{C}_c$ with $\mathbf{C}$, $\mathbf{C}_c$ constant

matrices. This requires the output of the plant and the output of the controller, respectively, to be linearly related to the state of the plant and the state of the controller. More generally, the magnitude of $\boldsymbol{\varepsilon}_{\mathbf{w}}$ is dependent on the distance between the centres, $(\tilde{\mathbf{x}}_i, \mathbf{r}_i)$, of the local models and the overlap in the validity functions, μ_i (similarly to $\boldsymbol{\varepsilon}_{\mathbf{w}^*}$ and $\boldsymbol{\varepsilon}_{\mathbf{y}^*}$). Clearly, the discussion in section 3.2 also applies to $\boldsymbol{\varepsilon}_{\mathbf{w}}$ and, when required, the magnitude of $\boldsymbol{\varepsilon}_{\mathbf{w}}$ can be kept sufficiently small that it can be neglected.

Observe that the dynamics of the closed-loop combination of the i^{th} individual plant local model and the i^{th} individual controller local model are

$$\dot{\mathbf{x}}_i = \overline{\mathbf{w}}_i, \quad \dot{\mathbf{x}}_{c_i} = \overline{\mathbf{w}}_{c_i}$$

$$\begin{bmatrix} \dot{\overline{\mathbf{w}}}_i \\ \dot{\overline{\mathbf{w}}}_{c_i} \end{bmatrix} = \begin{bmatrix} \phi_x^i & \phi_r^i \gamma_{x_c}^i \\ \phi_{c_i}^i \gamma_x^i & \phi_{x_c}^i \end{bmatrix} \begin{bmatrix} \overline{\mathbf{w}}_i \\ \overline{\mathbf{w}}_{c_i} \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_{c_i}^i \end{bmatrix} \mathbf{r}_{ref} \qquad (23)$$

$$\dot{\overline{\mathbf{y}}}_i = \begin{bmatrix} \gamma_x^i & \gamma_r^i \gamma_{x_c}^i \end{bmatrix} \begin{bmatrix} \overline{\mathbf{w}}_i \\ \overline{\mathbf{w}}_{c_i} \end{bmatrix}$$

It is evident that, provided the perturbation term, $\boldsymbol{\varepsilon}_{\mathbf{w}}$, is sufficiently small, the closed-loop dynamics of the blended multiple model system are described simply by the weighted combination of the closed-loop local models. Owing to this very direct relationship, and the linearity of the local models, the velocity-based blended multiple model approach clearly offers the potential to facilitate controller design.

3.4 Realisation of velocity-based blended multiple model systems

Some care is required when realising the blended multiple model system, (20), since the input is $\dot{\mathbf{r}}$ rather than $\mathbf{r}$. When the blended multiple model satisfies an integrability condition it may, by integrating, be reformulated as a nonlinear system of the form, (7) (Leith & Leithead 1997, 1998a). The input to the reformulated system is $\mathbf{r}$ but the terms in the resulting integrated system are related to the terms in the blended multiple model system by an integral equation. Alternatively, when the local models contain integral action, the blended multiple model system can be realised in velocity-based form (Leith & Leithead 1997, 1998a). This has the distinct advantage that the elements in the realisation are related *directly* to (20). In the context of control systems, pure integral action is frequently encountered. In the context of system identification and modelling, the plant under consideration can be readily augmented with integral action. The realisation of velocity-based systems, including situations where integral action is not present, is discussed in detail in Leith & Leithead (1997, 1998a)

4. Conclusion

The blending in a blended multiple model system is required to provide smooth interpolation, in some sense, between the local models with the aim of achieving an accurate representation with only a small number of local models. It is required that dynamics of the blended multiple model system are directly related to the local models. However, it is shown that the conventional type of blended multiple model system does not meet this requirement. Furthermore, although widely employed in the literature, it is noted that affine local models do not provide continuity with established linear methods and, in particular, do not possess the superposition property which is fundamental to linear systems. The magnitude of the inhomogeneous term in an affine local model may be extremely large and can strongly influence the solution. When the inhomogeneous term varies from one local model to another, it introduces an implicit feedback from the state and/or input of the overall blended multiple model system. Hence, the inhomogeneous term cannot, in general, simply be treated as a disturbance or small error.

A novel class of velocity-based blended multiple model systems is proposed which resolves these difficulties and for which

- the dynamics are directly related to the local models: the solution to a velocity-based blended multiple model system, locally to a specific operating point, is described by the solution to the linear system obtained by "freezing" the blended multiple model system at the relevant operating point. The resulting frozen system is simply a weighted linear combination of the local models.
- the solution to the blended multiple model system, locally to a specific operating point, is approximated by the weighted linear combination of the solutions to the local models.
- the local models are *linear*, thereby providing a degree of continuity with established linear methods.
- when combined with a corresponding velocity-based blended multiple model controller, the closed-loop dynamics are approximately described by the weighted combination of the closed-loop local models, facilitating design and analysis.

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