Modelling of Integrated Systems for Modular Linearisation-Based Analysis And Design

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Abstract
The selection of an appropriate representation for linearisation-based analysis and design of integrated systems is considered. It is shown that, in contrast to conventional linearisation-based representations, a velocity-based representation supports the modular analysis and design methodologies required with complex integrated systems.

1. Introduction

In response to increasingly stringent performance requirements, the trend in a wide range of engineering systems is towards tighter integration of the elements constituting the system. Whilst this frequently leads to increased dynamic interaction between sub-systems, the design of a complex system in an efficient and flexible manner necessitates a modular methodology whereby each sub-system has a well-defined interface to the rest of the system which is insensitive to the implementation details of the system. Such an approach enables the detailed design and implementation of each sub-system to be carried out separately and this is particularly important in projects where sub-contractors are involved.

There is, of course, a corresponding requirement for a representation which supports the analysis and design of systems in a modular fashion. Depending on the task at hand, a number of different representations are typically utilised for the analysis and design of complex systems; for example, Bond Graphs, Ordinary Differential Equations and Transfer Functions. Each representation possesses particular advantages and disadvantages which make it more, or less, suitable for a particular purpose. Traditionally, linearisation-based representations are often employed for the analysis and design of nonlinear systems. The system is approximated by a suitable linear system which, in the vicinity of an equilibrium operating point, exhibits similar dynamic characteristics (typically, the series expansion linearisation about the equilibrium operating point). Whilst conventional linearisation-based analysis is only valid locally to a specific equilibrium operating point, it has the considerable advantage that it maintains continuity with established linear analysis techniques for which a substantial body of experience has been accumulated. In order to determine appropriate linear approximations to a particular sub-system, knowledge of the equilibrium operating points is required. However, specifying *a priori* the relationship between the equilibrium inputs and outputs of a sub-system imposes, in general, a very strong restriction on the characteristics of the sub-system. For example, consider the nonlinear system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$. The functions, $\mathbf{F}(\bullet,\bullet)$ and $\mathbf{G}(\bullet,\bullet)$, are not independent since, $\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0$, $\mathbf{G}(\mathbf{x_o},\mathbf{r_o}) = \mathbf{y_o}$ must be jointly satisfied for all equilibrium input-output pairs $(\mathbf{x_o}, \mathbf{r_o})$. Indeed, when $\mathbf{F}(\bullet,\bullet)$ is invertible so that the equilibrium state, $\mathbf{x_o}$, is determined by $\mathbf{r_o}$, it follows that the output function, $\mathbf{G}(\bullet,\bullet)$, is, essentially, completely specified by the equilibrium input-output relationship and the choice of $\mathbf{F}(\bullet,\bullet)$. Consequently, in practice, a flexible equilibrium specification is required. However, since the equilibrium operating points are not those of the isolated sub-system but rather those of the sub-system when it is embedded in the overall system, the equilibrium operating points of a sub-system are, in general, strongly influenced by the characteristics of the overall integrated system and may change considerably as a result of even relatively small changes in other sub-systems. Hence, conventional linearisation-based representations do *not* readily support modular analysis and design approaches.

Whilst, in the context of integrated systems, the foregoing is perhaps the primary deficiency of conventional linearisation-based representations, it should be noted that there are also a number of other difficulties with such techniques. Firstly, the representations are only accurate in the vicinity of an equilibrium operating point whilst the requirement is usually to design a system which functions well not only when operating in the vicinity of a single equilibrium point but also during transitions between equilibrium operating points and periods of sustained non-equilibrium operation. Conventionally, this requirement is addressed by employing extensive simulation studies to iteratively refine the design, but this quickly becomes extremely time-consuming and inefficient for any but the simplest nonlinear systems. There is, therefore, a considerable incentive to directly incorporate, into the analytical part of the design procedure, knowledge of the plant dynamics during transitions between equilibrium operating points and during sustained non-equilibrium operation. Secondly, a number of distinct linear representations are typically employed during analysis and design (Leith & Leithead 1998a) which are not equivalent and which can make it difficult to incorporate insight obtained from the analysis into the design procedure. Thirdly, and at a more practical level, the determination of the equilibrium operating point is time-consuming and highly non-trivial for a complex

nonlinear system. Similarly, the numerical differentiation associated with conventional numerical linearisation about an equilibrium point is an undesirable ill-conditioned operation.

The velocity-based representation recently proposed by Leith & Leithead (1998a) rigorously generalises and extends the conventional series expansion linearisation at an equilibrium operating point and associates a linear system with every operating point, not just equilibrium operating points. An alternative description of a nonlinear system in terms of a family of linear systems is thereby established; namely, the velocity-based linearisation family. It is emphasised that the velocity-based formulation involves no loss of information and, in particular, does not involve either an inherent slow variation restriction nor is it confined to equilibrium operating points alone but instead encompasses every operating point, including those far from equilibrium. The velocity-based approach thereby relaxes the restriction to near equilibrium operation whilst maintaining the continuity with linear methods which is a principle advantage of the conventional linearisation-based analysis techniques. The velocity-based representation therefore resolves many of the deficiencies of conventional linearisation-based representations. However, the literature on the velocity-based representation is confined to "monolithic" systems for which there is no requirement to consider a decomposition into component sub-systems. The aim of this paper is, therefore, to investigate the application of the velocity-based representation to integrated systems and, in particular, consider the support, if any, provided by this representation for modular analysis and design.

2. Velocity-based representation

Before considering integrated systems, the velocity-based analysis and design representation is briefly summarised. Consider a nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{1}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable nonlinear functions with Lipschitz continuous first derivatives and $\mathbf{r}\in\Re^m$ denotes the input to the plant, $\mathbf{y}\in\Re^p$ the output and $\mathbf{x}\in\Re^n$ the states. The set of equilibrium operating points of the nonlinear system, (1), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which

$$\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0 \tag{2}$$

and the equilibrium output is

$$\mathbf{y_o} = \mathbf{G}(\mathbf{x_o}, \mathbf{r_o}) \tag{3}$$

Let $\Phi:\Re^n\times\Re^m$ denote the space consisting of the union of the states, $\mathbf{x}$, with the inputs, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear system, (1), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the system to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

The solution $\hat{\mathbf{x}}$ to the linear system (the "velocity-based linearisation")

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}}\,,\ \dot{\hat{\mathbf{w}}} = \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \tag{4}$$

$$\dot{\hat{\mathbf{y}}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \tag{5}$$

approximates the solution $\mathbf{x}$ to the nonlinear system locally to the operating point $(\mathbf{x_1}, \mathbf{r_1})$. Since a linear system (4)-(5) is associated with every operating point of the nonlinear system, there is a family of velocity-based linearisations associated with the nonlinear system. Whilst the solution to a single velocity-based linearisation is only a local approximation to the solution of the nonlinear system, the solutions to the members of this family can be pieced together to obtain an arbitrarily accurately *global* approximation to the solution of the nonlinear system. The relationship between the nonlinear system and its velocity-based linearisation, (4)-(5), is direct. Differentiating (1), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w},\ \dot{\mathbf{w}} = \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{6}$$

$$\dot{\mathbf{y}} = \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{7}$$

Dynamically, the velocity-form (6)-(7), with appropriate initial conditions, and (1) are equivalent (have the same solution, $\mathbf{x}$). Clearly, the velocity-based linearisation, (4)-(5), is simply the frozen form of (6)-(7) at the operating point, $(\mathbf{x_1}, \mathbf{r_1})$. Hence, numerical differentiation is not required in order to linearise a system in velocity-based form; instead, the velocity-based linearisation is obtained by simply "freezing" the velocity form of the nonlinear system.

The velocity-based formulation, (6)-(7), is of higher-order than the direct formulation, (1). However, reformulate (1), as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\boldsymbol{\rho}),\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\boldsymbol{\rho}) \tag{8}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are nonlinear functions and $\boldsymbol{\rho}(\mathbf{x},\mathbf{r})\in\Re^q$, $q\le m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ functions of $\boldsymbol{\rho}$ alone. Trivially, this reformulation can always be achieved by letting $\boldsymbol{\rho} = [\mathbf{x}^T\ \mathbf{r}^T]^T$, in which case $q=m+n$. The nonlinearity of the system is, however, frequently dependent on only a subset of the elements of the state and input, in which case the dimension, q, of the scheduling variable $\boldsymbol{\rho}$ is less than m+n. Assume, without loss of generality, that $\nabla_{\mathbf{x}}\boldsymbol{\rho}$ and $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ are constant. The velocity-form of the system, (8) , is

$$\dot{\mathbf{x}} = \mathbf{w} \tag{9}$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho})\,\nabla_{\mathbf{x}}\boldsymbol{\rho}\,)\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho})\,\nabla_{\mathbf{r}}\boldsymbol{\rho}\,)\,\dot{\mathbf{r}} \tag{10}$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\boldsymbol{\rho})\,\nabla_{\mathbf{x}}\boldsymbol{\rho}\,)\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\boldsymbol{\rho})\,\nabla_{\mathbf{r}}\boldsymbol{\rho}\,)\,\dot{\mathbf{r}} \tag{11}$$

which, as (10) and (11) depend only on $\mathbf{w}$, $\boldsymbol{\rho}$ and $\dot{\mathbf{r}}$, may be reformulated, equivalently, as

$$\dot{\boldsymbol{\rho}} = \nabla_{\mathbf{x}}\boldsymbol{\rho}\ \mathbf{w} + \nabla_{\mathbf{r}}\boldsymbol{\rho}\ \dot{\mathbf{r}} \tag{12}$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\boldsymbol{\rho})\,\nabla_{\mathbf{x}}\boldsymbol{\rho}\,)\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\boldsymbol{\rho})\,\nabla_{\mathbf{r}}\boldsymbol{\rho}\,)\,\dot{\mathbf{r}} \tag{13}$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\boldsymbol{\rho})\,\nabla_{\mathbf{x}}\boldsymbol{\rho}\,)\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\boldsymbol{\rho})\,\nabla_{\mathbf{r}}\boldsymbol{\rho}\,)\,\dot{\mathbf{r}} \tag{14}$$

Since the dimension of the scheduling variable $\boldsymbol{\rho}$ is, typically, much lower than that of $\mathbf{x}$, the order of (12)-(14) is usually only slightly higher than the order of (1).

There exists a rigorous, and direct, relationship between the dynamic characteristics of a nonlinear system and those of its velocity-based linearisation family. It is emphasised that the velocity-based linearisation family embodies the entire dynamics of the nonlinear system, (1),

with no loss of information and provides an alternative representation of the nonlinear system. There is no restriction to near equilibrium operation nor any slow variation requirement. Whilst the velocity-based representation is equivalent to the direct representation, (1) , in the sense that they each embody the entire dynamics of the nonlinear system, they are not necessarily equivalent with respect to other considerations. In particular, the direct relationship between the velocity-form of the nonlinear system and the velocity-based linearisation family and the linearity of the members of the latter family provides continuity with established linear theory which, for example, facilitates analysis (Leith & Leithead 1998a) and design (Leith & Leithead 1998b,c).

With regard to design, it is noted that the velocity-based representation provides direct support for divide and conquer design approaches, such as the gain-scheduling design methodology, whereby the design of a nonlinear system is decomposed into the design of an associated family of linear systems (Leith & Leithead 1998b,c). For example, since a velocity-based linearisation family is associated with a nonlinear plant, a corresponding linear controller family can be obtained by designing a linear controller for each member of the plant family. A nonlinear controller may then be determined for which the velocity-based linearisation family is the designed linear controller family. This approach resolves many of the deficiencies of the conventional gain-scheduling approach including the restriction to an excessively small neighbourhood of the equilibrium operating points. By allowing information about the plant dynamics at non-equilibrium operating points to be directly incorporated into the controller design, both sustained non-equilibrium operation and dynamic transitions between equilibrium operating points, including those which take the system far from equilibrium, can be accommodated. In particular, the velocity-based gain-scheduling approach can be employed to determine a dynamic inversion controller which achieves *global* linear dynamics when combined with the nonlinear plant (Leith & Leithead 1998c). Of course, the velocity-based gain-scheduling approach is quite general and also directly supports the design of feedback configurations for which the closed-loop dynamics are nonlinear.

3. Integrated systems

The velocity-based representation resolves many of the deficiencies of conventional linearisation-based analysis and design techniques. However, the existing literature on the velocity-based representation is confined to "monolithic" systems for which there is no requirement to consider a decomposition into component sub-systems. With regard to integrated systems, the velocity-based representation, in contrast to conventional linearisation-based representations, is not restricted to near equilibrium operation and does *not* require the equilibrium operating point of a system to be determined before linearisation-based analysis is possible. Rather, application of velocity-based analysis and design techniques only involves the much weaker requirement that the largest operating envelope of a system is known. As noted previously, it is non-trivial to determine the equilibrium operating point of a sub-system when it is embedded in the overall system and, furthermore, the equilibrium operating point is, in general, strongly influenced by the characteristics of the overall integrated system. Since the requirement to determine an equilibrium operating point is relaxed in the velocity-based framework, this framework resolves one of the principal difficulties, in the context of integrated systems, with conventional linearisation-based techniques. Of course, whilst this is necessary in order to de-couple the analysis and design of the sub-systems, it is not sufficient to support modular analysis and design. It is, in addition, also required that the analysis and design results obtained with a specific sub-system can be integrated in a direct and transparent manner with those obtained for other sub-systems. In order to investigate the integration of analysis and design results obtained for different sub-systems, it is sufficient to consider the series, parallel and feedback combination of sub-systems since these are the principal classes of interconnection in widespread use.

3.1 Series combination

Consider the nonlinear system

$$\dot{\mathbf{x}}_1 = \mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1), \qquad \mathbf{y}_1 = \mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \tag{15}$$

for which the velocity-based form is

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{w}_1 \\
\dot{\mathbf{w}}_1 &= \nabla_{\mathbf{x}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\mathbf{w}_1 + \nabla_{\mathbf{z}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\,\dot{\mathbf{z}}_1 \\
&\quad + \nabla_{\mathbf{r}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\,\dot{\mathbf{r}}_1 \\
\dot{\mathbf{y}}_1 &= \nabla_{\mathbf{x}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\mathbf{w}_1 + \nabla_{\mathbf{z}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\,\dot{\mathbf{z}}_1 \\
&\quad + \nabla_{\mathbf{r}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\,\dot{\mathbf{r}}_1
\end{aligned} \tag{16}$$

and the nonlinear system

$$\dot{\mathbf{x}}_2 = \mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2), \qquad \mathbf{y}_2 = \mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \tag{17}$$

for which the velocity-based form is

$$\begin{aligned}
\dot{\mathbf{x}}_2 &= \mathbf{w}_2 \\
\dot{\mathbf{w}}_2 &= \nabla_{\mathbf{x}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\mathbf{w}_2 + \nabla_{\mathbf{z}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\,\dot{\mathbf{z}}_2 \\
&\quad + \nabla_{\mathbf{r}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\,\dot{\mathbf{r}}_2 \\
\dot{\mathbf{y}}_2 &= \nabla_{\mathbf{x}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\mathbf{w}_2 + \nabla_{\mathbf{z}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\,\dot{\mathbf{z}}_2 + \\
&\quad + \nabla_{\mathbf{r}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\,\dot{\mathbf{r}}_2
\end{aligned} \tag{18}$$

The systems, (15) and (17), are cascaded together by setting $\mathbf{z}_2=\mathbf{y}_1$. The resulting system is

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z}), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z}) \tag{19}$$

where

$$\begin{aligned}
&\mathbf{x} = \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},\ \mathbf{r} = \begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix},\ \mathbf{z} = \mathbf{z}_1, \mathbf{y} = \mathbf{y}_2 \\
&\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z}) = \begin{bmatrix}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\\ \mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1))\end{bmatrix},\ \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z}) = \mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1))
\end{aligned} \tag{20}$$

for which the velocity-based form is

$$\dot{\mathbf{x}} = \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}$$

$$\dot{\mathbf{w}} = \begin{bmatrix} \dot{\mathbf{w}}_1 \\ \dot{\mathbf{w}}_2 \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \nabla_{\mathbf{z}_1}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{x}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \nabla_{\mathbf{x}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \begin{bmatrix} \nabla_{\mathbf{z}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \\ \nabla_{\mathbf{z}_1}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{z}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \end{bmatrix} \dot{\mathbf{z}}_1 + \begin{bmatrix} \nabla_{\mathbf{r}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \nabla_{\mathbf{z}_1}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{r}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \nabla_{\mathbf{r}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \end{bmatrix} \quad (21)$$

$$\dot{\mathbf{y}} = \dot{\mathbf{y}}_2 = \nabla_{\mathbf{z}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{x}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \quad \nabla_{\mathbf{x}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \nabla_{\mathbf{z}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{z}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)\dot{\mathbf{z}}_1 + \nabla_{\mathbf{z}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{r}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \quad \nabla_{\mathbf{r}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \end{bmatrix}$$

$$\mathbf{z}_2 = \mathbf{y}_1 = \mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1)$$

Evidently, (21) is just the system obtained when the systems, (16) and (18), are cascaded together. It follows that the velocity-based form of a system consisting of two cascaded sub-systems is identical to the system obtained by cascading together the velocity-based forms of the two sub-systems.

3.2 Parallel combination

The systems, (15) and (17), are combined in parallel by setting $\mathbf{z}_2=\mathbf{z}_1$. The resulting system is

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z}), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z}) \quad (22)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}, \quad \mathbf{z} = \mathbf{z}_1, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$$

$$\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z}) = \begin{bmatrix} \mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \\ \mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix}, \quad \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z}) = \begin{bmatrix} \mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \\ \mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix} \quad (23)$$

for which the velocity-based form

$$\dot{\mathbf{x}} = \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}$$

$$\dot{\mathbf{w}} = \begin{bmatrix} \dot{\mathbf{w}}_1 \\ \dot{\mathbf{w}}_2 \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \nabla_{\mathbf{z}_1}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{x}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \nabla_{\mathbf{x}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \begin{bmatrix} \nabla_{\mathbf{z}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \\ \nabla_{\mathbf{z}_1}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_2)\nabla_{\mathbf{z}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \end{bmatrix} \dot{\mathbf{z}}_1 + \begin{bmatrix} \nabla_{\mathbf{r}_1}\mathbf{F}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{r}_2}\mathbf{F}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \end{bmatrix} \quad (24)$$

is

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{x}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \begin{bmatrix} \nabla_{\mathbf{z}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) \\ \nabla_{\mathbf{z}_1}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix} \dot{\mathbf{z}}_1 + \begin{bmatrix} \nabla_{\mathbf{r}_1}\mathbf{G}_1(\mathbf{x}_1,\mathbf{r}_1,\mathbf{z}_1) & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{r}_2}\mathbf{G}_2(\mathbf{x}_2,\mathbf{r}_2,\mathbf{z}_1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{r}}_2 \end{bmatrix}$$

Evidently, (21) is just the system obtained when the systems, (16) and (18), are combined in parallel. Hence, the velocity-based form of a system consisting of the parallel combination of two sub-systems is identical to the system obtained by combining in parallel the velocity-based forms of the two sub-systems.

3.3 Feedback combination

Consider the nonlinear system with inputs, $\mathbf{r}$ and $\mathbf{z}$,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z}), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z}) \quad (25)$$

for which the corresponding velocity-based form is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{w} \\ \dot{\mathbf{w}} &= \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\mathbf{w} + \nabla_{\mathbf{z}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{z}} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{r}} \\ \dot{\mathbf{y}} &= \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\mathbf{w} + \nabla_{\mathbf{z}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{z}} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{r}} \end{aligned} \quad (26)$$

Assuming that

$$\mathbf{y}=\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{y}) \quad (27)$$

has a suitable solution

$$\mathbf{y}=\mathbf{N}(\mathbf{x},\mathbf{r}) \quad (28)$$

the system, (25), is enclosed in a feedback loop by setting $\mathbf{z}=\mathbf{y}$. The resulting closed-loop system is

$$\dot{\mathbf{x}} = \mathbf{M}(\mathbf{x},\mathbf{r}), \quad \mathbf{y} = \mathbf{N}(\mathbf{x},\mathbf{r}) \quad (29)$$

with

$$\mathbf{M}(\mathbf{x},\mathbf{r}) = \mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) \quad (30)$$

The velocity-based form of (29) is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{w} \\ \dot{\mathbf{w}} &= \nabla_{\mathbf{x}}\mathbf{M}(\mathbf{x},\mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{M}(\mathbf{x},\mathbf{r})\,\dot{\mathbf{r}} \\ \dot{\mathbf{y}} &= \nabla_{\mathbf{x}}\mathbf{N}(\mathbf{x},\mathbf{r})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{N}(\mathbf{x},\mathbf{r})\,\dot{\mathbf{r}} \end{aligned} \quad (31)$$

Combining (27) and (28)

$$\mathbf{N}(\mathbf{x},\mathbf{r}) = \mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) \quad (32)$$

Hence,

$$\begin{aligned} \nabla_{\mathbf{x}}\mathbf{M}(\mathbf{x},\mathbf{r}) &= \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) + \nabla_{\mathbf{z}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r}))\nabla_{\mathbf{x}}\mathbf{N}(\mathbf{x},\mathbf{r}) \\ \nabla_{\mathbf{r}}\mathbf{M}(\mathbf{x},\mathbf{r}) &= \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) + \nabla_{\mathbf{z}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r}))\nabla_{\mathbf{r}}\mathbf{N}(\mathbf{x},\mathbf{r}) \\ \nabla_{\mathbf{x}}\mathbf{N}(\mathbf{x},\mathbf{r}) &= \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) + \nabla_{\mathbf{z}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r}))\nabla_{\mathbf{x}}\mathbf{N}(\mathbf{x},\mathbf{r}) \\ \nabla_{\mathbf{r}}\mathbf{N}(\mathbf{x},\mathbf{r}) &= \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r})) + \nabla_{\mathbf{z}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{N}(\mathbf{x},\mathbf{r}))\nabla_{\mathbf{r}}\mathbf{N}(\mathbf{x},\mathbf{r}) \end{aligned} \quad (33)$$

and, by substituting (33) into(31), the closed-loop system, (29), can be directly reformulated as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{w} \\ \dot{\mathbf{w}} &= \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\mathbf{w} + \nabla_{\mathbf{z}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{z}} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{r}} \\ \dot{\mathbf{y}} &= \nabla_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\mathbf{w} + \nabla_{\mathbf{z}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{z}} + \nabla_{\mathbf{r}}\mathbf{G}(\mathbf{x},\mathbf{r},\mathbf{z})\,\dot{\mathbf{r}} \\ \mathbf{z} &= \mathbf{y} = \mathbf{N}(\mathbf{x},\mathbf{r}) \end{aligned} \quad (34)$$

Since $\mathbf{N}(\mathbf{x},\mathbf{r})$ satisfies (32), it is clear that (34) is the system obtained when the system, (26), is enclosed in a feedback loop by setting $\mathbf{z}=\mathbf{y}$. Consequently, the velocity-based form of the closed-loop system is identical to the system obtained by enclosing the velocity-based form of the open-loop system in a feedback loop.

3.4 Example

Consider two nonlinear dynamic systems

$$\dot{x}_1 = F_1(x_1,r_1) \quad (35)$$

$$\dot{x}_2 = F_2(x_2,r_2) \quad (36)$$

where

$$F_1(x_1,r_1) = -0.1x_1^3 - x_1 + 10.0 + r_1, \quad F_2(x_2,r_2) = -0.1x_2 - 0.1\tanh(x_2) - 165.0r_2^2 + 10.0r_2 \quad (37)$$

The unforced equilibrium state of (35) is 3.930 and that of (36) is 0.0. The corresponding equilibrium linearisations are

$$\delta\dot{x}_1 = -5.6335\,\delta x_1 + \delta r_1$$
$$\hat{x}_1 = \delta x_1 + 3.930, \quad \delta r_1 = r_1 - 0.0 \tag{38}$$

and

$$\delta\dot{x}_2 = -0.20\,\delta x_2 + 10.0\,\delta r_2$$
$$\hat{x}_2 = \delta x_2 + 0.0, \quad \delta r_2 = r_2 - 0.0 \tag{39}$$

However, suppose that the systems are coupled together with r_1 equal to x_2 and r_2 equal to $-x_1$. The equilibrium value of (x_1, x_2) is now (-0.1178, -10.1180) and the corresponding equilibrium linearisations are

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}-1.0042 & 1.0000\\ -1.8764 & -0.1000\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} \tag{40}$$
$$\hat{x}_1 = \delta x_1 - 0.1178, \quad \hat{x}_2 = \delta x_2 - 10.1180$$

In comparison, the dynamics obtained by coupling together the equilibrium linearisations (38) and (39),

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}-5.6335 & 1.0000\\ -10.000 & -0.2000\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} \tag{41}$$
$$\hat{x}_1 = \delta x_1 + 3.930, \quad \hat{x}_2 = \delta x_2 - 0.0$$

are quite different from the linear dynamics, (40); for example, the eigenvalues of (40) are (4.8406, -5.9447) whilst those of (41) are (-2.9168±j1.6184). This arises because the unforced equilibrium operating point changes when the systems are coupled together. In order to apply conventional linearisation-based analysis to a particular sub-system, it is, therefore, necessary to determine the equilibrium operating point of the complete coupled system. This is highly non-trivial for complex nonlinear systems (unlike the simple example presented here). Furthermore, the equilibrium operating point is, in general, influenced by the characteristics of the every sub-system. Consequently, conventional linearisation-based analysis techniques do *not* readily support a modular methodology.

The nonlinear systems, (35) and (36), may be reformulated as the velocity-based forms

$$\dot{x}_1 = w_1, \quad \dot{w}_1 = \left(-0.3x_1^2 - 1.0\right)w_1 + \dot{r}_1 \tag{42}$$

$$\dot{x}_2 = w_2, \quad \dot{w}_2 = \left(-0.2 + 0.1\tanh(x_2)^2\right)w_2 + \left(330.0r_2 + 10.0\right) \tag{43}$$

The velocity-based linearisation families of (35) and (36) are defined, respectively, by the frozen forms of (42) and (43). The velocity-form of the nonlinear system obtained when the two systems are coupled together with r_1 equal to x_2 and r_2 equal to $-x_1$ is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{w}_1\\ \dot{w}_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1.0 & 0\\ 0 & 0 & 0 & 1.0\\ -0.3x_1^2 - 1.0 & 0 & 0 & 0\\ -330.0x_1 - 10.0 & -0.2 + 0.1\tanh(x_2)^2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ w_1\\ w_2\end{bmatrix} \tag{44}$$

and the members of the velocity-based linearisation family of the coupled system consist of the frozen forms of (44) (or, equivalently, are obtained by coupling the relevant members of the velocity-based linearisation families of the individual nonlinear systems (35) and (36)). The velocity-based linearisation associated with an operating point indicates the dynamics of the nonlinear system at that operating point. The impulse responses of the velocity-based linearisations associated with a number of operating points of coupled nonlinear systems are depicted in figure 1: the variation of the dynamic characteristics with the operating point is clearly evident.

The velocity-based representation does not require an equilibrium operating point to be determined in order to analyse a system; rather, application of velocity-based analysis and design techniques only involves the much weaker requirement that the largest operating envelope of a system is known. In contrast to conventional linearisation-based representations, the velocity-based representation therefore supports modular linearisation-based analysis/design whereby the analysis/design of a sub-system is, as far as possible, insensitive to the internal details of the other sub-systems constituting an integrated system. Furthermore, the foregoing analysis establishes that the velocity-based linearisation families of the series, parallel and feedback combination of two nonlinear systems are identical to the series, parallel and feedback combination of the members of the velocity-based linearisation families of the individual nonlinear systems. Hence, in addition to de-coupling the analysis/design of sub-systems, analysis and design results utilising the velocity-based representation of a specific sub-system can be integrated in a direct and transparent manner with those obtained for other sub-systems.

4. Conclusions

It is shown that the velocity-based representation supports modular linearisation-based analysis and design of integrated systems In particular, and in contrast to conventional linearisation-based representations, the velocity-based representation

- does not require an equilibrium operating point to be determined in order to analyse a system; rather, application of velocity-based analysis and design techniques only involves the much weaker requirement that the largest operating envelope of a system is known. Trimming, which is highly non-trivial for complex nonlinear systems, is not required.
- does not require numerical differentiation; rather the velocity-based linearisation is obtained by simply "freezing" the velocity form of the nonlinear system.
- is not confined to near equilibrium operation but rather accommodates both transitions between equilibrium operating points and sustained non-equilibrium operation.
- provides a unified framework for analysis and design which employs a single linearisation, namely the velocity-based linearisation.

Furthermore, the velocity-based representations of the series, parallel and feedback combination of two nonlinear systems are identical to the series, parallel and feedback combination of the velocity-based representations of the individual nonlinear systems. Hence, in addition to de-coupling the analysis/design of sub-systems, analysis and design results utilising the velocity-based representation of

a specific sub-system can be integrated in a direct and transparent manner with those obtained for other sub-systems. The velocity-based representation therefore resolves many of the difficulties associated with conventional linearisation-based representations and provides direct support for the modular analysis and design methodologies required with complex integrated systems.

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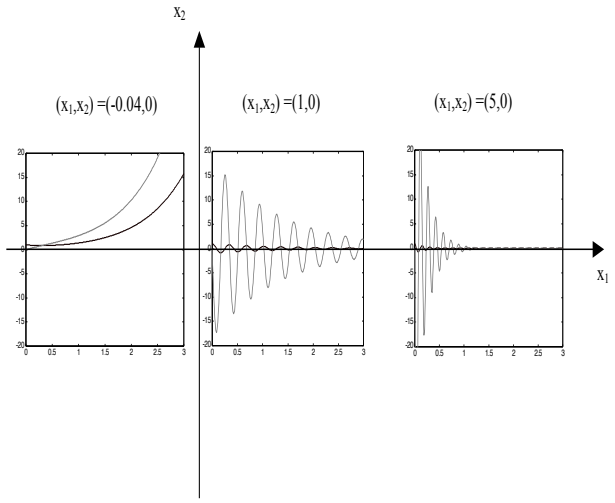

Figure 1 Impulse responses of the velocity-based linearisations associated with a number of operating points. It should be noted that the dynamics are strongly dependent on the value of x_1 but quite insensitive to the value of x_2. Responses are, therefore, only depicted for operating points at which x_2 is equal to zero.

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