1. In the Chinese Appetizer problem n people are eating n different appetizers arranged on a circular, rotating table. Someone spins the tray so that each person receives a random appetizer. What is the probability that everyone gets the same appetizer as before ? How does this compare with the bound obtained using Markov's inequality ?

Solution

(i) There are n equally likely orientations of the tray after it stops spinning. Everyone gets the right appetizer in exactly one of these n orientations. Therefore the probability is 1/n.

(ii) Let X be the number of people that get the right appetizer. In one orientation all n people get the right appetizer and X=n. For the other n-1 orientations X=0. Therefore E[X] = 1. By Markov's inequality,

$P(X \geq n) \leq E[X]/n = 1/n$

So in this example Markov's inequality is exact.

2. Consider two games. In game A, each time we play we win €2 with probability 2/3 and lose €1 with probability /3. In game B, each time we play we win €1002 with probability 2/3 and lose €2001 with probability 1/3. What is the expected winnings in both games ? What is the variance ? Using Chebyshev's inequality, compute an upper bound on the probability that after playing 10 rounds of each game the winnings deviate by more that ±10 from the expected value. Write a Matlab simulation to estimate the probability that make a loss after 10 rounds of each game.

Solution

Let X be the winnings.

Game A.

$E[X] = 2x2/3 - 1x1/3 = 1.$

$Var(X)=E[X^2]-E[X]^2=4x2/3+(-1)^2x1/3-1=2$

Game B.

$E[X] = 1002x2/3 – 2001x1/3 = 1.$

$Var(X)=E[X^2]-E[X]^2=1002^2x2/3+(-2001)^2x1/3-1 = 2004002$

So both games have the same expected winning, namely €1, but very different variance.

Using Chebyshev, let Y be the winnings after 10 rounds of a game. E[Y]=10E[X], Var(Y)=10Var(X). So,

Game A: $P(|Y-10| \geq 10) \leq 20/100=0.2$

Game B: $P(|X-10|\geq 10)\leq 20040020/100=200400$. We know the probability can be no more than 1, so this says no more than the trivial fact that $P(|X-10|\geq 10)\leq 1$.

Matlab:

```
P=[]; N=100000;

for i=1:N,

    A=0; B=0;

    for j=1:10,

        r=rand; A = A+2*(r<=2/3)-1*(r>2/3);

        r=rand; B = B+1002*(r<=2/3)-2001*(r>2/3);

    end

    P=[P; A B];

end

sum(P(:,1)<0)/N, sum(P(:,2)<0)/N
```

Probability of a loss after 10 rounds is about 0.02 in game A and about 0.44 in game B.

3. In a poll n turkeys selected independently at random are asked whether they vote for Christmas or not. Let X be the number of yes votes in our sample and use $S=X/n$ as our estimate of the actual fraction of turkeys who like Christmas. Let Let $X_i=1$ when the i^{th} turkey likes Christmas and 0 otherwise and assume $X_i\sim Ber(p)$. Using Chebyshev's inequality, how big should n be to ensure that this estimate is within 0.04 of the true fraction at least 95% of the time ?

<u>Solution</u>

Let $X_i=1$ is the i^{th} turkey likes Christmas and 0 otherwise. Then $E[X_i]=1xp+0x(1-p)=p$ and $Var(X_i) = E[X_i^2]-E[X_i]^2=1xp+0x(1-p)-p^2=p(1-p)$.

Therefore $E[X]=np$, $Var(X)=np(1-p)$ and $E[S]=E[X/n]=E[X]/n=p$, $Var(S)=Var(X/n)=Var(X)/n^2=p(1-p)/n$. Since $0\leq p\leq 1$ then $0\leq p(1-p) \leq 1/4$ and so $Var(S) \leq 1/4n$.

By Chebyshev's inequality,

$P(|S-p|\geq 0.04)\leq 1/(4x0.04^2n)$

and we require the RHS to be less than 0.05 (so that $|S-p|\leq 0.04$ holds 95% of the time). That is we require $1/(4x0.04^2n) \leq 0.05$ i.e. $n \geq 1/(4x0.04^2 \text{ x } 0.05)=3125$.