Overview

- Notation
- Sample Spaces
- Events
- Set Operations
- Axioms of Probability
Notation

- $\mathbb{Z}$ is the integers
- $\mathbb{R}$ is the real numbers
- $\{\cdots\}$ is a set
- $A \subseteq B$ means set $A$ is a subset of set $B$
- $A \in B$ means $A$ is a member of set $B$
- $\emptyset$ is the empty set
- $|A|$ is the number of elements in set $A$
- $|\text{such that }\}$
  e.g. $\{2z|z \in \{1, 2, 3\}\} = \{2, 4, 6\}$
- $P(E)$ means the probability of event $E$, although $\text{Prob}(E)$, $\mathbb{P}(E)$ can also be used.
Sample Spaces

Sample space $S$: the set of all possible outcomes of an experiment.

- Coin flip: $\{\text{Heads, Tails}\}$
- Flipping two coins: $\{(H, H), (H, T), (T, H), (T, T)\}$
- Roll of 6-sided die: $\{1, 2, 3, 4, 5, 6\}$
- Weather today: $\{\text{Sunny, Rainy, Snowy, Windy}\}$
- Number of emails in a day: $\{z | z \in \mathbb{Z}, z \geq 0\}$
- YouTube hours in a day: $\{z | z \in \mathbb{R}, 0 \leq z \leq 24\}$
Events

Event $E$: a subset of sample space $S$, $E \subset S$. A set of possible outcomes when an experiment is performed

- Coin comes up heads $\{\text{Heads}\}$
- One head and one tail on two flips $\{(H, T), (T, H)\}$
- Die roll is less than 3 $\{1, 2\}$
- Weather is wet $\{\text{Rainy, Snowy}\}$
- Number of emails is less than 20 $\{z|z \in \mathbb{Z}, 0 \leq z \leq 20\}$
- Wasted day (at least 5 hours on YT) $\{z|z \in \mathbb{R}, 5 \leq z \leq 24\}$
Set Operations on Events

Suppose $E$ and $F$ are events in sample space $S$ i.e. $E, F \subset S$
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$E \cup F$ is the event consisting of all the outcomes in $E$ or $F$. $\cup$ is called the union.
Set Operations on Events

Suppose $E$ and $F$ are events in sample space $S$ i.e. $E, F \subset S$.

$E \cap F$ is the event consisting of all the outcomes in both $E$ and $F$. $\cap$ is called the intersection.
Set Operations on Events

Suppose $E$ and $F$ are events in sample space $S$, $E, F \subset S$.

$E^c$ is the event consisting of all the outcomes not in $E$. $E^c$ is called the complement.
Set Operations on Events

Suppose $E$, $F$, $G$ are sets. Basic properties of set union and intersection:

- $E \cup F = F \cup E$ and $E \cap F = F \cap E$
- $(E \cup F) \cup G = E \cup (F \cup G)$ and $(E \cap F) \cap G = E \cap (F \cap G)$
- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$ and $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$
Set Operations on Events

Suppose $E$ and $F$ are events in sample space $S$ i.e $E, F \subset S$

$$(E \cup F)^c = E^c \cap F^c \quad (E \cap F)^c = E^c \cup F^c$$
(DeMorgan’s Laws)

$$( \bigcup_{i=1}^{n} E_i )^c = \bigcap_{i=1}^{n} E_i^c \quad ( \bigcap_{i=1}^{n} E_i )^c = \bigcup_{i=1}^{n} E_i^c$$
Axioms for Events

If $E$ and $F$ are events then so are:

- $E \cup F$
- $E \cap F$
- $E^c$ and $F^c$

Consequently:

- For events $E_i$, $i = 1, 2, \cdots n$ then
  - $E_1 \cup E_2$ is an event
  - $(E_1 \cup E_2) \cup E_3 = E_1 \cup E_2 \cup E_3$ is an event
  - $\bigcup_{i=1}^{n} E_i$ is an event
  - $\bigcap_{i=1}^{n} E_i$ is an event
- $S$ is an event since $S = E \cup E^c$ for any event $E$.
- The empty set $\emptyset$ is an event since $S^c = \emptyset$.
- Axioms really needed for sets with infinite number of elements (so $n$ could be infinite). A technicality, but we’ll confine ourselves to intersections, unions and complements when talking about events.
Axioms of Probability

Think of

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

where \( n(E) \) is the number of times event \( E \) occurs in \( n \) trials (we’ll come back to this later).
What basic properties does this quantity always have ?
Axioms of Probability

Axiom 1  $0 \leq P(E) \leq 1$

Axiom 2  $P(S) = 1$, where $S$ is sample space (set of all possible outcomes)

Axiom 3  If $E$ and $F$ are mutually exclusive ($E \cap F = \emptyset$) then $P(E \cup F) = P(E) + P(F)$. More generally,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

provided $E_i \cap E_j = \emptyset$ whenever $i \neq j$. 
Implications of Axioms

\[ P(E^c) = 1 - P(E) \]

- Since \( S = E \cup E^c \) and \( E \cap E^c = \emptyset \) then \( P(S) = 1 = P(E) + P(E^c) \)

E.g. \( S = \{ \text{Sunny, Rainy, Snowy, Windy} \} \)
- If \( E = \{ \text{Sunny} \} \) then \( E^c = \{ \text{Rainy, Snowy, Windy} \} \)
- \( P(E) + P(E^c) = 1 \) so \( P(E) = 1 - P(E^c) \)
- \( P(\{ \text{Sunny} \}) = 1 - P(\{ \text{Rainy, Snowy, Windy} \}) \)

Note \( P(S) + P(S^c) = P(S) + P(\emptyset) = 1 \) and \( P(S) = 1 \), so \( P(\emptyset) = 0 \) i.e. emptyset is just a formality.
Implications of Axioms

\[ E \subset F \text{ implies that } P(E) \leq P(F) \]

- Since \( F = E \cup (E^c \cap F) \) and \( E \cap E^c = \emptyset \) then
  \[ P(F) = P(E) + P(E^c \cap F) \]
- \( P(E^c \cap F) \geq 0 \) so \( P(E) = P(F) - P(E^c \cap F) \leq P(F) \)

E.g. \( S = \{\text{Sunny, Rainy, Snowy, Windy}\} \)
- If \( E = \{\text{Rainy}\} \) then \( F = \{\text{Rainy, Snowy}\} \)
- \( P(\{\text{Rainy}\}) \leq P(\{\text{Rainy, Snowy}\}) \)
Implications of Axioms

\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]

- \( E \cup F = E \cup (E^c \cap F) \) and \( E \cap (E^c \cap F) = \emptyset \) (mutually exclusive)
- \( F = (E \cap F) \cup (E^c \cap F) \), also mutually exclusive
- So \( P(E \cup F) = P(E) + P(E^c \cap F) \)
- and \( P(F) = P(E \cap F) + P(E^c \cap F) \) i.e. \( P(E^c \cap F) = P(F) - P(E \cap F) \)

E.g. \( S = \{\text{Sunny, Rainy, Snowy, Windy}\} \)
- If \( E = \{\text{Rainy}\} \) then \( F = \{\text{Snowy}\} \)
- \( P(\{\text{Rainy, Snowy}\}) = P(\{\text{Rainy}\}) + P(\{\text{Snowy}\}) - P(\{\text{Rainy}\} \text{ and } \{\text{Snowy}\}) \)
Equally Likely Outcomes

In some experiments all outcomes are equally likely. E.g. tossing a fair coin:

- \( S = \{ \text{Heads}, \text{Tails} \} \)
- \( P(\{\text{Heads}\}) = P(\{\text{Tails}\}) = p \) (coin is fair).
- Using axioms:
  - \( P(S) = P(\{\text{Heads}, \text{Tails}\}) = 1 \)
  - \( P(\{\text{Heads}, \text{Tails}\}) = P(\{\text{Heads}\}) + P(\{\text{Tails}\}) = 2p = 1 \).
  Solve to get \( p = \frac{1}{2} \).
Equally Likely Outcomes

Another example:

- $S = \{1, 2, \cdots, N\}$ and $P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$.
- Then $P(\{1\}) = \frac{1}{N}$, $P(\{2\}) = \frac{1}{N}$ etc

And for events consisting of multiple outcomes:

- $P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S} = \frac{|E|}{|S|}$
- E.g. $S = \{1, 2, 3, 4, 5, 6\}$ and $E = \{3, 4\}$ then $P(E) = \frac{2}{6}$. 
Rolling Two Dice

Roll two 6-sided dice

- What is the probability that the dice sum to 7?

And for events consisting of multiple outcomes:

- Sample space \( S = \{(1, 1), (1, 2), (1, 3), \cdots , (6, 5), (6, 6)\} \)
- Event \( E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \)
- \( P(E) = \frac{6}{36} = \frac{1}{6} \)
Tossing a Coin

Toss a fair coin twice:

- What is the probability that get two heads?
- Sample space $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- Event $E = \{(H, H)\}$
- $P(E) = \frac{1}{4}$
- What is the probability that get heads then tails?
- Event $E = \{(H, T)\}$
- $P(E) = \frac{1}{4}$
- What is the probability that get one head and one tail?
- Event $E = \{(H, T), (T, H)\}$
- $P(E) = \frac{2}{4} = 0.5$
Drawing Balls from a Bag

Have an bag containing 4 red balls and 3 white balls. Draw 3 balls.

• What is \( P(1 \text{ red ball and 2 white balls drawn}) \) ?

• Can draw 3 balls out of bag containing 7 balls in \( \binom{7}{3} = 35 \) ways. So sample space \( S \) is of size \( |S| = 35 \).

• Event \( E \) is of size \( \binom{4}{1} \binom{3}{2} = 12 \)

• \( P(1 \text{ red ball and 2 white balls drawn}) = \frac{12}{35} \)

• What is \( P(\geq 2 \text{ red balls drawn}) \) ?

• Event \( E \) is of size \( \binom{4}{2} \binom{3}{1} + \binom{4}{3} = 22 \), \( P(\geq 2 \text{ red balls drawn}) = \frac{22}{35} \)

• What is \( P(\geq 2 \text{ white balls drawn}) \) ?

• Event \( E \) is of size \( \binom{3}{2} \binom{4}{1} + \binom{3}{3} = 13 \), \( P(\geq 2 \text{ white balls drawn}) = \frac{13}{35} \)
Important Trick

Often its hard to count the number of times an event $E$ occurs, but easy to count the number of time event $E$ does not occur. Use $P(E) = 1 - P(E^c)$, where $E^c$ is the event that $E$ does not occur.

• We flip a coin 3 times. What is the probability that there is at least one heads?
  • Sample space $|S| = 2^3 = 8$.
  • Event that no heads is $E^c = \{(T, T, T)\}$. $|E^c| = 1$ so $P(E^c) = \frac{1}{8}$.
  • Therefore $P(E) = 1 - P(E^c) = 1 - \frac{1}{8}$ is the probability of one or more heads.
  • What if we flipped the coin 10 times? 100 times?

• We toss a dice twice. What is the probability that the sum is greater than 3?
  • Sample space $|S| = 6^2 = 36$.
  • Event that less than or equal to three is $E^c = \{(1,1), (1,2), (2,1)\}$. $|E^c| = 3$ so $P(E^c) = \frac{3}{36}$.
  • Therefore $P(E) = 1 - P(E^c) = 1 - \frac{3}{36}$ is the probability the sum is greater than 3.
Birthdays

What is the probability of event $E$ that of $n$ people two or more share the same birthday (regardless of year)?
Birthdays

What is the probability event $E$ that of $n$ people one or more of them shares a birthday with you? Let's ask the complement $E^c$: of $n$ people what is the probability that none of them shares a birthday with you?

- $|S| = 365^n$
- $|E^c| = 364^n$
- $P(E^c) = \frac{364^n}{365^n}$
- $P(E) = 1 - P(E^c)$.

Some values:
- When $n = 23$ then $P$ (no matching birthdays) $\approx 0.94$
- When $n = 75$ then $P$ (no matching birthdays) $\approx 0.81$
- When $n = 100$ then $P$ (no matching birthdays) $\approx 0.76$

Why are these probabilities so much higher than before?
Poker Hands

• Straight flush is 5 consecutive cards of same suit.
• What is \( P(\text{straight flush}) \) ?
• Sample space \(|S| = \binom{52}{5} = 2598960\)
• 4 suits. For each suit (each with 13 cards) can get a straight flush 10 different ways. Event \(|E| = 10 \times 4\).
• \( P(\text{straight flush}) = \frac{40}{2598960} \approx 1.5 \times 10^{-5} \)
• What is \( P(\text{four of a kind}) \) ?
• 13 ways to select 4 cards of the same kind. 5th card can be selected from remaining 12 kinds, and from each of 4 suits i.e. \( 12 \times 4 \) ways. Event \(|E| = 13 \times 12 \times 4 = 624\).
• \( P(\text{four of a kind}) = \frac{624}{2598960} \approx 12.4 \times 10^{-4} \)