ST3009 Mid-Term Test 2019

Attempt **all** questions. Time: 1 hour 30 mins.

1. A (large) bag contains 4 white and 3 black sheep

a) Suppose four sheep are drawn from the bag without replacement. What is the probability that they are alternately of different colours? [5 marks]

Suppose now that each sheep is put back into the bag after being drawn.

b) What is the probability that the four sheep are alternately of different colours now? [5 marks]

c) Let X be the number of sheep drawn from the bag that are white. Calculate:
   (i) E[X] [5 marks]
   (ii) Var(X) [5 marks]

d) State the definition of statistical independence for random variables. [5 marks]

e) Let Y be the number of sheep drawn from the bag that are black. Are random variables X and Y independent? Explain your answer using the definition of statistical independence. [5 marks]

Model Solution

1(a) There are two sequences with alternating colours, namely BWBW and WBWB. The probability of sequence BWBW is now (3/7)(4/6)(2/5)(3/4). The probability of sequence WBWB is (4/7)(3/6)(3/5)(2/4). The overall probability (3/7)(4/6)(2/5)(3/4)+ (4/7)(3/6)(3/5)(2/4)=0.1714

(b) The probability of sequence BWBW is $(3/7)^2(4/7)^2$ and the probability of sequence WBWB is the same. So the overall probability is $2(3/7)^2(4/7)^2=0.12$

(c) $P(X=1)=4(4/7)(1-4/7)^3$. $P(X=2)=6(4/7)^2(1-4/7)^2$. $P(X=3)=4(4/7)^3(1-4/7)$. $P(X=4)=(4/7)^4$. We then have:

$E[X]=0\times P(X=0)+1\times P(X=1)+2\times P(X=2)+3\times P(X=3)+4\times P(X=4) = 2.28$

and

$Var(X)=E[X^2]-E[X]^2$ with $E[X^2]=1^2\times P(X=1)+2^2\times P(X=2)+3^2\times P(X=3)+4^2\times P(X=4)=6.20$ so $Var(X)=0.979$.

An alternative solution for those who noticed that X is a binomial random variable with n=4 and p=4/7 is to use the fast that a Bin(n,p) random variable has mean np and variance np(1-p).

(d) Two random variables X and Y are independent if

$P(X=x \text{ and } Y=y)=P(X=x)P(Y=y)$

for all pairs of values (x,y) that the RVs can take.

(e) Y=4-X. Since Y depends on X it is not independent. We can confirm this formally using the definition of independence by finding values x and y such that $P(X=x \text{ and } Y=y)\neq P(X=x)P(Y=y)$. For example, choosing x=4 and y=4 then $P(X=x \text{ and } Y=4)=0$ and $P(X=4)= (4/7)^4$, $P(Y=4)=P(X=0)= (3/7)^4$ and we are done.

2.

a) Define what is meant by the terms "sample space", "random event" and "random variable". Give examples of each. [5 marks]

b) Consider two random variables X and Y that take values in set {0,1}. Starting from the definition of the expectation and variance of a random variable show that

(i) $E[aX]=aE[X]$ [5 marks]

(ii) $Var(aX)=a^2Var(X)$ [5 marks]

c) Let X be a random variable which takes value 1 when a user clicks on a displayed advert, and 0 otherwise.

(i) How can we interpret E[X] and why might its value be of interest to an advertiser? When might the value of E[X] be less useful (hint: think about the variance)? [5 marks]

ii) Let Y be a random variable which takes value 1 when a user has visited web site https://Y.com. Suppose Prob(X=1)=0.001, Prob(Y=1|X=1)=0.1 and Prob(Y=1|X=0)=0.01. Using Bayes Rule calculate Prob(X=1|Y=1). Given an advertiser has knowledge of the value of Prob(X=1|Y=1) vs Prob(X=1) how might a user visiting https://Y.com affect whether the advertiser displays an advert to that user or not ? [5 marks]

Model Solution

2a) The sample space is the set of possible outcomes of an experiment, a random event is a subset of the sample space, a random variable is a function mapping from the sample space to a real number. For example, the sample space of a coin flip might be {Heads, Tails}, a random event {Tails} and a random variable might take value 1 when the outcome is tails and 0 otherwise.

b)

(i) $E[aX] = a\times1\times Prob(X=1) + a\times0\times Prob(X=0) = a\,(1\times Prob(X=1) + 0\times Prob(X=0)) = a\,E[X]$ since $E[X]= 1\times Prob(X=1)+ 0\times Prob(X=0)$

(ii) $Var(aX)= E[(aX)^2] – E[aX]^2$ and $Var(X)=E[X^2]-E[X]^2$. Using (i), $E[aX]^2=(aE[X])^2=a^2E[X]^2$. $E[(aX)^2] = (a\times1)^2\times Prob(X=1) +(a\times0)^2\times Prob(X=0)= a^2\,(1^2\times Prob(X=1) +0^2\times Prob(X=0)) = a^2\,E[X^2]$. So $Var(aX)= a^2\,E[X^2]-a^2E[X]^2=a^2(E[X^2]-E[X]^2)$_$=a^2Var(X)$.

c)

(i) If we think about the setup where an advert is displayed to a user N times, where N is large, then N×E[X] is the number of times that the user clicks on the advert. Suppose the advertiser makes €0.01 for each click, then 0.01× N×E[X] is the revenue from the N displays of the advert. This might be less useful when the variance of X is large, since then the revenue from N displays of the advert is highly variable and might be far from E[X].

(ii) Prob(X=1|Y=1) = Prob(Y=1|X=1)P(X=1)/P(Y=1)= Prob(Y=1|X=1)P(X=1)/(Prob(Y=1|X=1)P(X=1)+ Prob(Y=1|X=0)P(X=0))=0.1×0.001/(0.1×0.001+0.01× (1-0.001))=0.099. Since Prob(X=1|Y=1)=0.099 is much larger than Prob(X=1)=0.001, a user is more likely to click on the advert if they have visited https://Y.com. To maximise revenue the advertiser should therefore be more likely to display an advert to users that it knows have visited https://Y.com.

3. Five people play the game of "odd-man-out" to determine who pays for a meal. In this game, each person flips a coin. If one person's coin comes up different to

all others (i.e. there is one H and four T's or there is one T and four H's), then that person pays. Otherwise, everyone flips again. They go on doing this until someone is chosen.

a) What is the probability that one person's coin comes up differently from the others at a given round? [5 marks]

b) What is the expected number of times they must flip before they know who should pay? Hint $\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$ [5 marks]

c) Explain how to generate a Bernoulli random variable (similar to a coin toss) using then rand() function in matlab. [5 marks]

d) Write pseudo-code for a short simulation of this game that outputs the number of flips made. [5 marks]

Model Solution

3(a) There are 2^5 =32 ways to flip 5 coins. In 10 of these one person is different from all the others (5 where one person gets heads and the rest tails and 5 where one person gets tails and the rest heads). So the probability is 10/32=0.3125

(b) Let X be the number of flips. Then E[X]=1×P(X=1)+ 2×P(X=2)+ 3×P(X=3)+… = 1×p+ 2×(1-p)p+ 3×$(1-p)^2$p+… where p=0.3125 is the probability of stopping. We can rewrite this as $E[X] = \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = 1/p$

(c) Use X=rand()<p where p is the probability that X=1.

(d)