ST3009 Mock Mid-Term Test

Attempt all questions. Time: 1 hour 30 mins.

1. (i) Define the terms "sample space", "event" and "random variable" and give an example of each. [10 points]

(ii) What is an indicator random variable and what is the probability mass function of a discrete random variable? [5 points]

(iii) Define the conditional probability of an event and state Bayes Theorem. [5 points]

(iv) Explain what is meant by "marginalization". [5 points]

Solution: See notes.

2. Suppose we have two bags, labeled A and B. Bag A contains 3 white balls and 1 black ball, bag B contains 1 white ball and 3 black balls. We toss a fair coin and select bag A if it comes up heads and otherwise bag B. From the selected bag we now draw 5 balls, one after another, replacing each ball in the bag after it has been selected (the bag always contains 4 balls each time a ball is drawn). We observe 4 white balls and 1 black ball. What is the probability that we selected bag A? Hint: use Bayes Rule. [20 points]

Solution: Let E be the event that choose bag A and E: the event that choose bag B. Let F be the event that we observe 4 white and 1 black balls. We need to calculate P(E|F). By Bayes Rule we know that P(E|F)=P(F|E)P(E)/P(F).

We know P(E)=½ and P(F|E)=\(\binom{5}{4}(\frac{3}{4})^4(\frac{1}{4})\) since the probability of drawing a white ball from bag A is \(\frac{3}{4}\) and a black ball is \(\frac{1}{4}\) and there are five different combinations possible (black ball drawn first, second and so on). So we just need P(F). We have:

\[
P(F)=P(F|E)P(E)+P(F|E^c)P(E^c) = \binom{5}{4}(\frac{3}{4})^4(\frac{1}{4})(\frac{1}{2})+ \binom{5}{1}(\frac{1}{4})^4(\frac{3}{4})(\frac{1}{2})
\]

since P(F|E^c)=\(\binom{5}{4}(\frac{1}{4})^4(\frac{3}{4})\) and P(E^c)=1-P(E)= \(\frac{1}{2}\). Therefore,

\[
P(E|F)= \frac{(\frac{3}{4})^4(\frac{1}{2})}{(\frac{3}{4})^4(\frac{1}{2})+ (\frac{1}{4})^4(\frac{3}{4})} = 0.96
\]

3. (i) Define the expected value of a random variable. Give a proof that the expected value is linear i.e. E[X+Y]=E[X]+E[Y] for random variables X and Y. [5 points]

(ii) Define what it means for two random variables to be independent. Give a proof that when two random variables X and Y are independent then E[XY]=E[X]E[Y]. [5 points]
(iii) Define the covariance and correlation of two random variables $X$ and $Y.$

\[ 5 \text{ points} \]

\textit{Solution:} See notes.

4. (i) A bag contains 30 balls, of which 10 are red and the other 20 blue. Suppose you take out 8 balls from this bag, with replacement. What is the probability that among the 8 balls in this sample exactly 3 are red and 5 are blue? \[ 5 \text{ points} \]

\textit{Solution.} Since we have replacement each draw is independent. Probability of a red is $10/30$ and of a blue is $20/30$. So probability of 3 red and 5 blue is $(10/30)^3(20/30)^5$.

(ii) Now suppose that the balls are taken out of the bag without replacement. What is the probability that out of 8 balls exactly 3 are red and 5 are blue? \[ 10 \text{ points} \]

\textit{Solution.} Since its without replacement the draws are no longer independent. The total number of ways to take 8 balls out of 30 is $\binom{30}{8}$. Picking 3 red balls of 10 can be done $\binom{10}{3}$ ways. Similarly, picking 5 blue balls of out 20 can be done $\binom{20}{5}$ ways. So the probability is $\binom{10}{3}\binom{20}{5}/\binom{30}{8}.$