**ST3009 Mock Mid-Term Test**

Attempt all questions. Time: 1 hour 30 mins.

1. (i) Define the terms “sample space”, “event” and “random variable” and give an example of each. [10 points]

(ii) What is an indicator random variable and what is the probability mass function of a discrete random variable? [5 points]

(iii) Define the conditional probability of an event and state Bayes Theorem. [5 points]

(iv) Explain what is meant by “marginalization”. [5 points]

2. Suppose we have two bags, labeled A and B. Bag A contains 3 white balls and 1 black ball, bag B contains 1 white ball and 3 black balls. We toss a fair coin and select bag A if it comes up heads and otherwise bag B. From the selected bag we now draw 5 balls, one after another, replacing each ball in the bag after it has been selected (the bag always contains 4 balls each time a ball is drawn). We observe 4 white balls and 1 black ball. What is the probability that we selected bag A? Hint: use Bayes Rule. [20 points]

3. (i) Define the expected value of a random variable. Give a proof that the expected value is linear i.e. \( E[X+Y]=E[X]+E[Y] \) for random variables \( X \) and \( Y \). [5 points]

(ii) Define what it means for two random variables to be independent. Give a proof that when two random variables \( X \) and \( Y \) are independent then \( E[XY]=E[X]E[Y] \). [5 points]

(iii) Define the covariance and correlation of two random variables \( X \) and \( Y \). [5 points]

4. You have a bag that contains \( b \) blue marbles and \( r \) red marbles. You choose \( k \leq b+r \) marbles at random (without replacement) and let \( X \) be the number of blue marbles in your sample. Now let us define the indicator random variables \( X_i \), \( i=1,2,...,k \) such that \( X_i=1 \) if the \( i \)'th chosen marble is blue and \( X_i=0 \) otherwise. Then, we can write: \( X=X_1+X_2+...+X_k \).

Using the above equation, show that

(i) \( E[X] = \frac{kb}{b+r} \) [5 points]

(ii) \( Var(X) = \frac{kbr}{(b+r)^2} \frac{b+r-k}{b+r-1} \) [10 points]