Notes 5: Game Theory:

Two person strictly competitive games of the following form:

You make a decision, opponent makes a decision and there is no co-operation.

Zero-sum games: means your winnings =opponents loss.

Your aim is to maximise your gain.

Example 1:

Original pay-off matrix

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>R2</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>R3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>R4</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

Removing dominated strategies:

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>R2</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>R3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>R4</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>10</td>
</tr>
</tbody>
</table>

R picks 1-4, C picks 1-4, R’s score is as given, C’s score is minus given, e.g., both pick 2, R gets 1 and C gets -1 (i.e., zero-sum game).

Choice 1 dominates choice 2 if all alternatives for 1 at least as good as 2 and some are better. Don’t play dominated strategies.

a) C4 dominated by C2, so delete C4.
b) Now R2 dominates R3, so delete R3.
c) Now C2 dominates C3, so delete C3.
d) Now R2 dominates R1 and R4, so play R2.
e) Now C plays C2.

So both R and C play 2, with R getting 1 and C getting -1. This is the value of the game.

Note: Neither R or C can raise value by playing second if opponent plays this choice.
Example 2:

Ray and Dottie choose a campsite. R wants to camp high, D wants to camp low. R chooses N-S, D chooses E-W and camp on intersection.

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>7</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>R2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>R3</td>
<td>5</td>
<td><em>3</em></td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>R4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

Dominated strategies deleted, no other.

Note: If R chooses 1 or 4, D might outguess him and choose 2, giving height of 2. If picks 3, guarantees at least 3. D might see this the other way around, pick 2 guaranteeing at most 3.

Pair (R3, D2) are in equilibrium. If either chooses from the pair, the other can do no better than choosing from the pair also.

Game has a value of 3 (which either side can force). Point which is minimum of row and maximum of column is called a saddle point (or equilibrium point).

So check for saddle points. If there is such a point then this is the value of the game.

Note: Game may have several saddle points, but they all have the same value.

Example 3:

<table>
<thead>
<tr>
<th></th>
<th>Clever Student</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>Even</td>
</tr>
<tr>
<td>Even</td>
<td>-1</td>
</tr>
<tr>
<td>Odd</td>
<td>+1</td>
</tr>
</tbody>
</table>

Protect against being out guessed each time by tossing a fair coin (Even if H, Odd if T). Win half the time and expected value of game is 0. Can even reveal strategy to opponent.

Can’t force a higher expected value as opponent can do the same, e.g., Odd (1/2), Even (1/2) is equilibrium strategy for each and value of game is 0.

Example 4:

<table>
<thead>
<tr>
<th></th>
<th>Moriarty</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Canterbury</td>
</tr>
<tr>
<td>Holmes</td>
<td>Canterbury</td>
</tr>
<tr>
<td>Dover</td>
<td>100 (Holmes escapes)</td>
</tr>
</tbody>
</table>

Suppose H chooses Dover with probability $p_H$, M chooses Dover with probability $p_M$. 
Expected payoff to H is:

\[ E = 0 \cdot P(HC \cap MC) + 0 \cdot P(HD \cap MD) + 100P(HD \cap MC) + 50P(HC \cap MD) \]

\[ = 50(1 - p_H)p_M + 100p_H(1 - p_M) = 50(p_M + 2p_H - 3p_Mp_H) \]

H chooses \( p_H \) to maximise this, M chooses \( p_M \) to minimise this.

Note: If \( p_H = \frac{1}{3} \), \( E = 50(\frac{1}{3} + 2\cdot\frac{1}{3} - \frac{2}{3}) = \frac{100}{3} \) regardless of \( p_M \), so H can force value 100/3

Similarly, if \( p_M = \frac{2}{3} \), \( E = 50\left(\frac{2}{3} + 2p_H - 2p_H\right) = \frac{100}{3} \) regardless of \( p_H \).

So either player can force a value 100/3 and we say \((p_H, p_M) = (\frac{1}{3}, \frac{2}{3})\) are equilibrium strategy (if either deviates the other can achieve a better outcome than 100/3).

In general: \( A = (a_{ij}), i = 1, ..., m, j = 1, ..., n \) is payoff matrix for game. R chooses one of the rows \( i = 1, ..., m \) X picks one of the columns \( j = 1, ..., n \). If R picks \( i \) and C picks \( j \) payoff \( a_{ij} \) to R and \(-a_{ij}\) to C, i.e., zero-sum.

Mixed strategy for R is choice \( p = (p_1, ..., p_m) \) where \( p_i \) is probability R chooses \( i \) (each \( p_i \geq 0 \) and \( \sum_{i=1}^{m} p_i = 1 \)), and for C is \( q = (q_1, ..., q_n) \) with \( q_i \geq 0 \) and \( \sum_{i=1}^{n} q_i = 1 \). So payoff \( a_{ij} \) is given with probability \( p_i q_j \).

Expected payoff to R is \( \sum_{i,j} p_i q_j a_{ij} = p^T A q \) and C gets minus this. R chooses \( p \) to make this big, while C chooses \( q \) to make this small.

Best that R can guarantee if C can ‘outguess’ R is \( V_R = \max_p (\min_q (p^T A q)) \).

Best that C can guarantee if R can ‘outguess’ C is \( V_C = \min_q (\max_p (p^T A q)) \).

**Theorem: Minimax for Two Person Zero-Sum Games:**

\( V_R = V_C = V \) (value of game) = \( p^* T A q^* \) for some choices \( p^*, q^* \) the equilibrium strategies for the game, where \( p^* T A q^* = \max_p (p^T A q) = \min_q (p^T A q) \).

Hence either player can force \( V \) and if one player plays their equilibrium strategy, the other player can do no better than playing their own equilibrium strategy (but could do worse).

Proof Omitted.

**Solving Games:**

To find value and minimax strategies of zero-sum games:

i) Delete dominated strategies.

ii) Check for saddle points, as these are solutions if they exist.

iii) In general use simplex algorithms to solve (we don’t cover that here), instead consider special 2xC case.
2 by 2 with no saddle point:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>a₁₁ a₁₂ p₁</td>
</tr>
<tr>
<td>2</td>
<td>a₂₁ a₂₂ p₂</td>
</tr>
<tr>
<td>q₁</td>
<td>q₂</td>
</tr>
</tbody>
</table>

Value of game if $p, q$ are minimax:

$$V = a_{11}p_1q_1 + a_{12}p_1q_2 + a_{21}p_2q_1 + a_{22}p_2q_2 = p_1(a_{11}q_1 + a_{12}q_2) + p_2(a_{21}q_1 + a_{22}q_2)$$

So for minimax, must have $a_{11}q_1 + a_{12}q_2 = a_{21}q_1 + a_{22}q_2 = V$.

As $q_1 + q_2 = 1$, this gives $q_1$, $q_2$, $V$, and solve for $p_1$, $p_2$ similarly.

**Graphical Solutions for 2 by n Games:**

$$A = \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{pmatrix},$$

R wants to maximise $\max_p (\min_q (p^T A q))$. Suppose R chooses R2 with probability $p$, R1 otherwise. R wants to maximise:

$$f(p) = \min_q (p^T A q) = \min_q (1-p, p) \begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \text{ with } q_1 \geq 0, \sum q_i = 1.$$

$$= \min_q (a_1(1-p) + pb_1, \ldots, a_n(1-p) + pb_n) \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

$$= \min_q (q_1(a_1(1-p) + pb_1), \ldots, q_n(a_n(1-p) + pb_n))$$

$$= \min(q_1(a_1(1-p) + pb_1), \ldots, q_n(a_n(1-p) + pb_n))$$
Example:

\[ A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix} \]

Solving intersection of C2 and C3, e.g., \(3 - 2p = 1 + 5p\) gives \(p = 2/7\), hence \(V = 17/7\), i.e., probability R plays 2 is 2/7. C either plays C2 or C3, found by solving 2 by 2 game:

\[
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
R & 1 & 3 & 1 \\
2 & 1 & 6 \\
\end{array}
\]

Knowing \(V = 17/7\), C plays \((0, \frac{5}{7}, \frac{2}{7}, 0)\).

Note: in 2 by \(n\) game there are always 2 columns that C can pick between.

**Non-Zero Sum Games (2 players):**

Suppose each player chooses as before, but payoff to R is not necessarily minus the payoff to C.

Example:

\[
\begin{array}{c|cc|c|c}
 & 1 & 2 & p & 1 - p \\
\hline
R & (2,1) & (0,0) & q & 1 - q \\
2 & (0,0) & (1,2) & \end{array}
\]

Here \((\cdot, \cdot)\) is payoff in utility to R and C respectively.
Mixed strategies: R plays 1 with probability $p$, 2 otherwise. C plays 1 with probability $q$, 2 otherwise. They determine $p, q$ independently.

**Equilibrium:**

Given one player plays their equilibrium strategy, the other can’t do better than plays theirs also. Nash showed such equilibrium strategies exist (uniquely).

**Example:**

Payoff to R is $2pq + (1 - q)(1 - p), p^* = 1/3$ cancels $q$ out.
Payoff to C is $pq + 2(1 - p)(1 - q), q^* = 2/3$ cancels $p$ out.

Play $p^* = \frac{1}{3}, q^* = \frac{2}{3}$. Payoff to R is $2 \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$ same as payoff to C.

But $p = q = 1/2$ is better for both.

**Example: Prisoner’s Dilemma**

<table>
<thead>
<tr>
<th>Victor</th>
<th>Hugo</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t Confess</td>
<td>(1,1)</td>
<td>(4,0)</td>
</tr>
<tr>
<td>Confess</td>
<td>(0,4)</td>
<td>(3,3)</td>
</tr>
</tbody>
</table>

V and H have committed a crime together. They should confess or not? Note interviewed independently. The entries in the table are years in prison for V and H, respectively. Note Loss= Utility. Should V confess? Should H confess?

Consider V’s situation:

If H confesse, then if V confesses he gets 3 years, better than not confessing (4 years).
If H doesn’t confess then if V confesses he gets 0 years, better than not confessing (1 year).

Hence confessing is always better and is a stationary dominating strategy, so here it is better for both to play worst individual strategy.

**Group Decision Making:**

**Example:**

Population wants to choose a political party. Each person gives preferences, e.g., $Y >^* R >^* B$. How do we combine all votes to get group preference?

Possible principle: Majority Rule, e.g., Group prefers $Y$ to $R$ if more than 50% of voters do.

**Example:**


Majority rule implies that for the group: $Y >^* R$ (2 to 1), $R >^* B$ (2 to 1), $B >^* Y$ (2 to 1).
So group is a money pump.

Basic problem of ‘social choice’ theory. Each individual in group has personal preference ranking over a collection of alternatives. How can we combine these into a group preference ranking ‘fairly’? For example, have a collection of rewards \( r_1, \ldots, r_k \) and group has \( m \) members. Member \( i \) has personal preference ranking \( \succeq^*_i \). Collection \( (\succeq^*_1, \ldots, \succeq^*_m) \) is preference profile of group.

Each personal preference ranking satisfies:

(D1) For each pair \( r_i, r_j \) for each profile \( \succeq^*_k \) we must have one and only one of \( r_i \succ^*_k r_j, r_j \succ^*_k r_i, \) or \( r_i \sim^*_k r_j \).

(D2) \( r_i \succ^*_k r_j, r_j \succ^*_k r_s \Rightarrow r_i \succ^*_k r_s \) etc. (transitivity) and \( r_i \sim^*_k r_j, r_j \sim^*_k r_s \Rightarrow r_i \sim^*_k r_s \) etc.

**Definition:**

A Social Welfare Function (SWF) is a function which operates on a collection of individual preference profiles each of which obeys (D1) and (D2), and yields a group or social ordering \( \succeq^*_g \) which also obeys (D1) and (D2).

Note: For small number of individuals, this is a group decision problem, e.g., each individual expresses cake preferences for a shared cake purchase. For many individuals, it is a social choice problem.

Arrow suggested four ‘reasonable’ conditions for a SWF:

(U) Unrestricted domain: \( \succeq^*_g \) is defined and obeys (D1), (D2) for any collection \( (\succeq^*_1, \ldots, \succeq^*_m) \) which all obey (D1) and (D2).

(D) No dictatorship: No individual \( i \) such that \( \succeq^*_i \) automatically becomes \( \succeq^*_g \).

(P) Pareto principle: If \( r_i \succ^*_k r_j \) for all \( k \), then \( r_i \succ^*_g r_j \).

(I) Independence of irrelevant alternatives: Suppose some rewards are deleted from the reward set. Then if no individual changes preferences between rewards that remain, the group preference between remaining alternatives doesn’t change.

**Comment on (I):** Suppose half people have \( B \succ^* Y \succ^* R \) and half have \( Y \succ^* R \succ^* B \).

Suppose \( R \) is no longer available, so half have \( B \succ^* Y \) and half have \( Y \succ^* B \), then might suggest \( Y \sim^* B \), but this ignores that \( Y \) was previously in everyone’s top 2 choices.

Compare second situation:

Half have \( B \succ^* New\ B \succ^* Y \) and half have \( Y \succ^* B \succ^* New\ B \) where \( B \) and \( New\ B \) are the same outcome just renamed, and now suppose \( New\ B \) becomes no longer available. Now have alternative conclusion, but preferences between \( Y \) and \( B \) are same in both situations.

Thus Arrow argues that it is wrong to view a preference for \( Y \) to \( B \) as stronger than \( B \) to \( Y \).
Arrow’s Impossibility Theorem:

Provided there are at least 3 possible rewards and at least 2 individuals, then there is no SWF which meets all of conditions (U), (D), (P) and (I).

Note: means that for any SWF, there are sets of preferences which break at least one condition. However, if only 2 rewards, then majority rule does satisfy all of (U), (D), (P) and (I).

Proof: Omitted.

Utilitarianism:

Social choices should attempt to maximise ‘well-being’ (happiness) of citizenry. Given a choice between social options, place at top the option which produces most pleasure overall, and at bottom, the option which produces least pleasure overall.

How do we measure for individual? How do we combine individual pleasures? E.g., sum or product etc?

A possible way of measuring individual pleasure is by utility. Bus how can the group combine utilities?

Suppose $m$ citizens and $r$ social choices $x = (x_1, ..., x_r)$. Each citizen $i$ individually rational and so has utility function on $x$, $u_i$. Scale each utility to lie in $[0,1]$, with 0 utility of worst outcome and 1 utility of best.

Example:

Council have enough money for exactly one of: Swimming Pool (S), Library (L), Car park (C), Museum (M), or Nothing (N). Town has 2 citizens:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>L</th>
<th>C</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>$u_2$</td>
<td>0.1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Introduce planner, knows citizens utilities and must plan on their behalf. P individually rational so has utility function $w$ on $x$. P wants to make fair decision for community.

Suppose P obeys following two conditions for ‘social rationality’:

(A) Anonymity: P doesn’t know which citizen made which vote. P would create the same $w$ utility for any permutation of votes $u_1, u_2, ...$

(SP) Strong Pareto: If each citizen indifferent between two rewards, then so is planner. If nobody prefers $a$ to $b$, some people prefer $b$ to $a$, the P prefers $b$ to $a$.

In example, P gets 2 votes for (S, L, C, M, N), namely $(1, ½, 0, ½, 0)$ and $(1/10, 0, 1, 0, 0)$.

P doesn’t know who made which vote.
Note $u_1(L) = u_1(M) \Rightarrow L \sim_1^* M$ and $u_2(L) = u_2(M) \Rightarrow L \sim_2^* M$ so $L \sim_1^* M$ by (SP).

Now construct, for each option, a vector of utilities $u_i(x_i) = (u_1(x_i), u_2(x_i), \ldots)$

E.g., $u(S) = \left(\frac{1}{2}, 0, 0, 0, 0\right)$, $u(L) = \left(\frac{1}{2}, 0, 0, 0, 0\right)$, $u(C) = \left(0, 1, 0, 0, 0\right)$, $u(M) = \left(\frac{1}{2}, 0, 0, 0, 0\right)$, $u(N) = \left(0, 0, 0, 0, 0\right)$

Note that if $u_1(x_j) = u_1(x_k)$ then $u_i(x_j) = u_i(x_k)$ for all $i$, so $x_j \sim_i^* x_k$ for all $i$, so $x_j \sim_1^* x_k$ by (SP).

So $w(x_j) = w(x_k)$ and so $w(x_j)$ must be a function of $u(x_j)$, e.g., $w(x_j) = f(u(x_j))$ for some function $f$.

**Theorem: Haranyi**

Group of individuals, collection of rewards. Individual $i$ specifies utilities $u_i(x)$ for each $x$ scaled to $[0,1]$. Planner wants to construct utility $w$ obeying (A) and (SP). Then the only choice for $w$ is $w(x) = u_1(x) + \cdots + u_m(x)$ for all $x$.

For example:

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>S</th>
<th>L</th>
<th>C</th>
<th>M</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$u_2$</td>
<td>0.1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$w = u_1 + u_2$</td>
<td>1.1</td>
<td>0.5</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Proof: Omitted.