**Notes 3: Multi-Attribute Utility:**

**Example:**

Journey choices: coach or train. Decide key attributes are cost and time. Need to construct utility over (cost, time) pairs. Start by giving a method for specifying utility over 2 attributes. Suppose reward \( r(x, y) \), want to specify \( u(r(x, y)) = u(x, y) \).

Special case: Utility Independence (UI) – \( X \) is UI of \( Y \) if preferences with varying values of \( X \) and fixed value of \( Y \) are independent of the fixed value chosen for \( Y \), e.g., for \((t, c)\) pairs:

Suppose \( \frac{1}{2}(3,20) + \frac{1}{2}(5,20) >^* (4,20) \) (*), then if \( t \) is UI of \( c \), then (*) holds for all ticket prices, e.g.,

\[
\frac{1}{2}(3,25) + \frac{1}{2}(5,25) >^* (4,25) \text{ (**).}
\]

If \( X \) is UI of \( Y \) what is the form of \( u(x, y) \)?

Fix \( Y \) at \( y_0 \) and specify \( u(x, y_0) \) which is utility over \( x \), i.e., gambles over pairs with \( Y = y_0 \).

Fix \( Y \) at \( y_1 \) and specify \( u(x, y_1) \) which is utility over \( x \), i.e., gambles over pairs with \( Y = y_1 \).

UI implies \( u(x, y_0), u(x, y_1) \) represent the same preferences for gambles on \( x \) (e.g., compare (*) with (**)).

Hence as utility is unique up to a positive linear transformation \( u(x, y_1) = \alpha u(x, y_0) + \beta \) \((\alpha > 0)\), where \( \alpha, \beta \) depend only on \( y_1, y_0 \) but not on \( x \).

Now keeping \( y_0 \) fixed and letting \( y_1 \) vary we prove the following lemma.

**Lemma:**

\( X \) is UI of \( Y \) if and only if, for any fixed \( y_0 \), \( u(x, y) = \alpha(y)u(x, y_0) + \beta(y) \) (with \( \alpha \) a positive function), \( \alpha, \beta \) depend on \( y_0 \) but not \( (x, y) \).

Similarly, if \( Y \) is UI of \( X \) then \( u(x, y) = \gamma(x)u(x_0, y) + \delta(x) \).

If \( X \) is UI of \( Y \) and \( Y \) is UI of \( X \), then say \( X \) and \( Y \) are Mutually Utility Independent (MUI).

**Theorem:**

\( X, Y \) MUI. Choose any pair \((x_0, y_0)\) and set \( u(x_0, y_0) = 0 \) (call \((x_0, y_0)\) origin of utility).

Then \( u(x, y) = u(x, y_0) + u(x_0, y) + ku(x, y_0)u(x_0, y) \) (where \( k \) is a constant to be specified).

**Proof:**

Suppose \( X, Y \) MUI and set \( u(x_0, y_0) = 0 \).

From lemma \( u(x, y) = \alpha(y)u(x, y_0) + \beta(y) \) \((1)\) and \( u(x, y) = \gamma(x)u(x_0, y) + \delta(x) \) \((2)\)

Set \( x = x_0 \) in \((1)\) so \( u(x_0, y) = \alpha(y)u(x_0, y_0) + \beta(y) = \alpha(y)0 + \beta(y) = \beta(y) \).
Similarly $y = y_0$ in (2) gives $u(x, y_0) = \delta(x)$.

Hence $\alpha(y)u(x, y_0) + u(x_0, y) = u(x, y) = \gamma(x)u(x_0, y) + u(x, y_0)$

Or $(\alpha(y) - 1)u(x, y_0) = (\gamma(x) - 1)u(x_0, y)$

Or $\frac{\alpha(y) - 1}{u(x_0, y)} = \frac{\gamma(x) - 1}{u(x, y_0)}$, but LHS is a function of $y$ only, while RHS is a function of $x$ only, so only way LHS=RHS for all $x,y$ is if they are a constant $k$.

So $\frac{\alpha(y) - 1}{u(x_0, y)} = \frac{\gamma(x) - 1}{u(x, y_0)} = k$ for all $x, y$

Hence $\gamma(x) = ku(x, y_0) + 1$

Implementing into (2) now gives $u(x, y) = u(x, y_0) + u(x_0, y) + ku(x,y_0)u(x_0, y)$ as needed.

How do we interpret $k$?

Assume MUI so $u(x, y) = u(x, y_0) + u(x_0, y) + ku(x,y_0)u(x_0, y)$

Then $u(G_1) - u(G_2) = \frac{k}{2} [u(x_2, y_0) - u(x_1, y_0)] [u(x_0, y_2) - u(x_0, y_1)]$ with each expression in $[\cdot]$ > 0 as $x_2 > x_1$ and $y_2 > y_1$.

We prefer $G_1$ to $G_2 \iff u(G_1) - u(G_2) > 0 \iff k > 0$.

We prefer $G_2$ to $G_1 \iff k < 0$, and $G_1 \sim^* G_2 \iff k = 0$.

So $k$ acts like a ‘risk aversion’ effect.

If $k > 0$ we sometimes say $X, Y$ are ‘complements’.

If $k < 0$ we sometimes say $X, Y$ are ‘substitutes’.

Alternative way of writing $u(x, y) = u(x, y_0) + u(x_0, y) + ku(x,y_0)u(x_0, y)$:

Either $k = 0$ and $u(x, y) = u(x, y_0) + u(x_0, y)$ (*),

or $k \neq 0$ and $1 + ku(x, y) = (1 + ku(x, y_0)) (1 + ku(x_0, y))$ (**).
So MUI function is either additive (*) or multiplicative (**). This generalises to \( n \) attributes \( x_1, \ldots, x_n \). If these are MUI then:

Either \( u(x_1, \ldots, x_n) = u(x_1) + \cdots + u(x_n) \), or \( 1 + ku(x_1, \ldots, x_n) = \prod_{i=1}^{n} (1 + u(x_i)) \)

Choosing \( x_{10}, \ldots, x_{n0} \) and setting \( u(x_{10}, \ldots, x_{n0}) = 0 \), then in the above

\[
u(x_i) = u(x_{10}, x_{20}, \ldots, x_{(i-1)0}, x_i, x_{(i+1)0}, \ldots, x_{n0}).
\]

\( x_1, \ldots, x_n \) are MUI if, for any division of \( X \) into two groups \( Y, Z \), your preferences over gambles in \( (Y, Z) \) with \( Y \) varying and \( Z \) fixed does not depend on the fixed choice for \( Z \).

**Mutually Utility Independent Hierarchy:**

Utility comprised of attributes (MUI). Each attribute comprised of sub-attributes (MUI). Each sub-attribute comprised of sub-sub-attributes (MUI) etc.

```
Overall utility \( X_O \)

Cost \( X_C \)  Benefit \( X_B \)

Financial \( X_F \)  Ethical \( X_{Et} \)

Financial \( X_F \)  Ethical \( X_{Et} \)

Environmental \( X_{En} \)  Social \( X_S \)

Environmental \( X_{En} \)  Social \( X_S \)
```

Specify marginal utilities at each ‘end node’ and joining together utilities using multiplicative and additive form using methods we now describe.

**Specifying Multi-Attribute Utilities:**

\( X, Y \) MUI. Want to separate questions about risk from questions about ‘trade-offs’ between \( X \) and \( Y \). Two methods:

1) More ‘risk’ questions.

2) More ‘trade-off’ questions.

First method:

i) Choose origin \((x_0, y_0)\) and set \( u(x_0, y_0) = 0 \). Can choose any values, but best to choose ones which simplifies the remainder.

E.g., journey cost and time are MUI attributes. Choose \( u(3,18) = 0, t_0 = 3, c_0 = 18 \).

ii) Fix \( y_0 \) and specify marginal utility \( u(x, y_0) \), fix \( x_0 \) and specify marginal \( u(x_0, y) \) (these are ‘risk’ questions).
E.g., suppose you decide you are risk neutral for money gambles in this range, and that you are risk neutral for time gambles in this range. Then with origin defined by $u(3,18) = 0$:

$$u(3, c) = \alpha c + \beta$$

and because of origin value we have $u(3, c) = \alpha(c - 18)$.

Similarly $u(t, 18) = \gamma(t - 3)$.

iii) $u(x, y_0)$ and $u(x_0, y)$ can be transformed by arbitrary positive linear transformation. So fix transform by choosing two values $x_1, y_1$ such that $(x_1, y_0) \sim^* (x_0, y_1)$.

If $(x_1, y_0) >^* (x_0, y_0)$ set $u(x_1, y_0) = u(x_0, y_1) = 1$.

If $(x_0, y_0) >^* (x_1, y_0)$ set $u(x_1, y_0) = u(x_0, y_1) = -1$.

This fixes constants in each marginal utility (this is a ‘trade-off’ question).

E.g., suppose judge $(3,32) \sim^* (5,5,18) <^* (18,3)$, then set $u(3,32) = u(5,5,18) = -1$, which gives

$$\alpha = -\frac{1}{14} \text{ and } \gamma = -\frac{2}{5} \text{ (*)}. \tag{1}$$

iv) $u(x, y) = u(x, y_0) + u(x_0, y) + ku(x, y_0)u(x_0, y)$. Find $k$ by choosing a final trade-off pair $(x_2, y_2) \sim^* (x_3, y_3)$ so $u(x_2, y_2) = u(x_3, y_3)$. So unless $u(x_3, y_0)u(x_0, y_0) = u(x_2, y_0)u(x_0, y_2)$ can solve for $k$ (otherwise find another pair).

E.g., suppose $(3,32) \sim^* (4,25)$

$u(4,25) = u(3,25) + u(4,18) + ku(3,25)u(4,18) = u(3,32) = -1$ by (*)

With $u(3,25) = -\frac{(25-18)}{14}, u(4,18) = \frac{-2(4-3)}{5}$ we find $k = -1/2$.

Hence $u(t, c) = -\frac{(c-18)}{14} - \frac{2(t-3)}{5} - \frac{(c-18)(2(t-5))}{2+14+5}$

**Specifying Multi-Attribute Utility Method 2:**

$X, Y \text{ MUI. Fix } u(x_0, y_0) = 0$. Fix $x_0$ and specify $u(x_0, y)$. Suppose you find it difficult to specify $u(x, y_0)$. Alternative is to specify an iso-preference curve:

Every point $(t, m)$ on curve is equally preferred by you.
So, for each $x$ choose $y(x)$ such that \((x, y(x)) \sim^* (x_0, y_0)\). E.g., \((t_0, m(t_0)) = (10 \text{ hr}, 100)\), then $t = 15, m(t) = 50$ means \((10, 100) \sim^* (15, 50)\).

**Theorem:**

\(X, Y \text{ MUI.} \) Iso-Preference curve \((x, y(x))\) through \((x_0, y_0)\): 
\[
u(x, y) = \frac{u(x_0, y) - u(x_0, y(x))}{1 + k u(x_0, y(x))}
\]

So to specify \(u(x, y)\):

i) Fix origin at \(u(x_0, y_0) = 0\)

ii) Specify marginal utility \(u(x_0, y)\) and impose constraint \(u(x_0, y_0) = 0\).

iii) Specify iso-preference curve \((x, y(x))\) through \((x_0, y_0)\).

iv) Find \(k\) by choosing a trade-off \((x_1, y_1) \sim^* (x_2, y_2) \sim^* (x_0, y_0)\) and solve \(u(x_1, y_1) = u(x_2, y_2)\).

**Proof:**

\[
u(x, y) = u(x, y_0) + u(x_0, y) + ku(x, y_0)u(x_0, y)\]

For all \(x, (x, y(x)) \sim^* (x_0, y_0)\).

Hence \((x, y(x)) = 0 \implies u(x, y_0) + u(x_0, y(x)) + ku(x, y_0)u(x_0, y(x)) = 0\)

Hence \(u(x, y_0) = \frac{-u(x_0, y(x))}{1 + ku(x_0, y(x))}\)

Rearranging gives result.

Note on specification method 2: In step ii) Specify \(u(x_0, y)\).

Suppose instead specify \(v(x_0, y) = au(x_0, y) + b. v(x_0, y_0) = 0 \implies b = 0\). Do we need to fix \(a\)?

No, as we would get \(v(x, y) = \frac{v(x_0, y) - v(x_0, y(x))}{1 + ku(x_0, y(x))} = \frac{au(x_0, y) - u(x_0, y(x))}{1 + ku(x_0, y(x))}\), i.e., same utility.

**The Use of Information in Decision Making:**

**Example:**

Oil company – should we drill? Well can be:

<table>
<thead>
<tr>
<th>Profit</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dry (w_1)</td>
<td>-$1m</td>
</tr>
<tr>
<td>Wet (w_2)</td>
<td>$1m</td>
</tr>
<tr>
<td>Soaking (w_3)</td>
<td>$5m</td>
</tr>
</tbody>
</table>

For cost \$0.3m we can take soundings which either give a good \((G)\) or not good \((\bar{G})\) result. Suppose:

<table>
<thead>
<tr>
<th></th>
<th>(G)</th>
<th>(\bar{G})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w_1)</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>(w_2)</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>(w_3)</td>
<td>0.8</td>
<td>0.2</td>
</tr>
</tbody>
</table>
i.e, \( P(G|w_3) = 0.8 \), so by theorem of total probability:
\[
P(G) = P(G|w_1)P(w_1) + P(G|w_2)P(w_2) + P(G|w_3)P(w_3) = 0.45 \Rightarrow P(\bar{G}) = 0.55
\]

Question? Should we take soundings? Should we drill? Aim: Maximise EMV

Here, for example,
\[
P(w_1|G) = \frac{P(G|w_1)P(w_1)}{P(G)} (\text{Bayes' Theorem}) = \frac{0.2 \times 0.5}{0.45} = 0.222
\]

Choice: No soundings and drill with EMV $1m. Soundings have some information (you would choose differently seeing \( G \) – drill, or \( \bar{G} \) – not drill), but too expensive. How much are they worth? If they were free, all values on lower branch increase by $0.3m, so value at chance node B is $1.05m. Compare with decision node A, e.g., $1m. Thus the value of the soundings is $1.05m-$1m=$0.05m