**Decision Analysis**

**Introductory Example**

Company produces product. Expect increased demand next year.

Decide between: \( d_1 \) - new machinery, \( d_2 \) – old machinery & overtime.

Uncertainty exists as to consequence of decision. In particular sales may be good or bad.

Payoff table:

<table>
<thead>
<tr>
<th></th>
<th>Bad Sales ((\theta_1))</th>
<th>Good Sales ((\theta_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>260</td>
<td>440</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>300</td>
<td>420</td>
</tr>
</tbody>
</table>

Columns: outcome set / set of uncertain states

Rows: decision set.

So, e.g., choose \( d_1 \) and get \( \theta_1 \) then consequence is 260.

Note: if sales good prefer \( d_1 \) to \( d_2 \), if sales bad prefer \( d_2 \) to \( d_1 \).

Need to consider ‘probability’ of good sales. Suppose judge \( P(\text{Good sales})=0.6, P(\text{Bad sales})=0.4 \).

Note: All probabilities in this course are subjective judgements of uncertainty of Decision Maker (DM). Usually need to carry out a sensitivity analysis on these subjective judgements.

Represent problem as decision tree:

- decision node (you choose)
- outcome or chance node (don’t choose)

![Decision Tree Diagram]

Choose \( d_1 \) or \( d_2 \)

How do we choose between \( d_1 \) and \( d_2 \)? One decision criteria is to choose decision with highest Expected Monetary Value (EMV).
Reminder: $X$ random quantity (r.q.), values $x_1, \ldots, x_k$ with probabilities $p_1, \ldots, p_k$.

$E[X] = p_1x_1 + \cdots + p_kx_k$

$E[X + Y] = E[X] + E[Y]$ ‘always’

$E[E[X|Y]] = E[X] = E[X|Y = y_1]P(Y = y_1) + \cdots E[X|Y = y_r]P(Y = y_r)$

Node A: $\text{EMV}(d_1) = 0.6 \times 440 + 0.4 \times 260 = 368$

Node B: $\text{EMV}(d_2) = 0.6 \times 420 + 0.4 \times 300 = 372$

$\text{EMV}(d_2) > \text{EMV}(d_1)$ so choose $d_2$ with EMV 372.

**Risk Profile of Chosen Decision:**

Collection of possible outcomes of your decision, with their probabilities.

In this case outcomes: 300 420

probability: 0.4 0.6

Question: Why use EMV criterion? When will it be useful? How can we do better? In particular, how do we handle non-monetary rewards?

**Example extended: 2 year horizon decisions:**

First year – new machine or overtime, uncertain states: good/bas sales.

Second year – If in year 1 choose new machinery, then in year 2, either choose more machinery or overtime. If in year 1 choose overtime, then must use in year 2.

Uncertain states: High, Medium, or Low sales.

Following probability judgements:

If sales bad in year 1: Year 2 $P(H)=0.4$, $P(M)=0.4$, $P(L)=0.2$

If sales good in year 1: Year 2 $P(H)=0.5$, $P(M)=0.4$, $P(L)=0.1$

These are conditional probabilities, e.g., $0.5 = P(H_2|G_1) = \frac{P(H_2 \cap G_1)}{P(G_1)} = \frac{P(G_1|H_2)P(H_2)}{P(G_1)}$
Draw the decision tree:

Can’t immediately choose between \( d_1 \) and \( d_2 \) as expected value of \( d_1 \) depends on later choices on tree. But if we reach 2a (e.g., choose \( d_1 \) and get good sales) we can choose between \( d_3 \) and \( d_4 \).

\[
\text{EMV}(d_3) = 0.5 \times 820 + 0.4 \times 790 + 0.1 \times 760 = 802
\]
\[
\text{EMV}(d_4) = 0.5 \times 850 + 0.4 \times 816 + 0.1 \times 790 = 830.4
\]
\[
\text{EMV}(d_4) > \text{EMV}(d_3) \text{ so if we reach 2a then choose } d_4. \text{ Similarly at 2b choose } d_6.
\]

What is EMV(\( d_1 \) )? Either Good sales (probability 0.6), EMV=830.4, or Bad sales (probability 0.4), EMV=660.

So \( \text{EMV}(d_1) = 0.6 \times 830.4 + 0.4 \times 660 = 762.2 \)
\[
(= E[\text{EMV}(d_1|Year 1 sales)] = \text{EMV}(d_1|Good)P(Good) + \text{EMV}(d_1|Bad)P(Bad)
\]

Similarly \( \text{EMV}(d_2)=711, \text{ so EMV}(d_1) > \text{EMV}(d_2) \text{ so choose } d_1 \text{ (and then if Good } d_4, \text{ or Bad } d_6 – \text{ both overtime) and overall EMV 762.2}


Build tree left to right. Starting at right of tree:

i) As each chance node is reached, mark with EMV of tree to right of node (and remove tree to right)

ii) As each decision node is reached, mark with EMV of branch to right that is highest (and remove tree to right)

iii) Repeat until we reach start node.
**Example Continued:**

![Decision Tree Diagram]

**Sensitivity Analysis**

How much would we need to change values on tree to change our decision? E.g., $P(\text{Good sales})=p$. How critical is choice $p=0.6$?

\[
EMV(d_1) = p \times 440 + (1 - p) \times 260 = 180p + 260 \\
EMV(d_2) = p \times 420 + (1 - p) \times 300 = 120p + 300
\]

$EMV(d_2) > EMV(d_1) \iff 120p + 300 > 180p + 260 \iff p < 2/3$, i.e., 7% change in $p$ is most we can make without changing decision.

We can make better decisions by first collecting information to reduce uncertainty and thus change our probabilities and thus change our decisions (maybe).

How much would you pay to foresee exactly the future?

If you know for sure that sales will be good, choose $d_1$, profit 440. If you know for sure that sales will be bad, choose $d_2$, profit 300.

$P(\text{good})=0.6$, so Expected Money Value Under Certainty (EMVUC) is: $0.6\times440+0.4\times300=384$ (where e.g., $0.6=P(\text{Good})$ and 440 is best payoff if we know good sales).

Expected Value of Perfect Information (EVPI) = EMVUC-EMV=384-372=12

i) Absolute max we would spend on market research.

ii) If EVPI is a large proportion of EMVUC then it suggests high benefit from market research.

Using advanced decision tree (2 year):

<table>
<thead>
<tr>
<th>Outcome (Y1)</th>
<th>Outcome (Y2)</th>
<th>Prob</th>
<th>Outcome for Decision</th>
<th>Max Payout</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>H</td>
<td>0.3</td>
<td>850</td>
<td>850</td>
</tr>
<tr>
<td>G</td>
<td>M</td>
<td>0.24</td>
<td>816</td>
<td>816</td>
</tr>
<tr>
<td>G</td>
<td>L</td>
<td>0.06</td>
<td>790</td>
<td>790</td>
</tr>
<tr>
<td>B</td>
<td>H</td>
<td>0.16</td>
<td>690</td>
<td>700</td>
</tr>
<tr>
<td>B</td>
<td>M</td>
<td>0.16</td>
<td>650</td>
<td>670</td>
</tr>
<tr>
<td>B</td>
<td>L</td>
<td>0.08</td>
<td>620</td>
<td>650</td>
</tr>
</tbody>
</table>

Risk profile of our chosen decision
Note, e.g., here $P(G_1 \cap H_2) = P(G_1)P(H_2|G_1) = 0.6 \times 0.5 = 0.3$

EMV of our decision = $0.3 \times 850 + \ldots + 0.08 \times 620 = 762.2$

EMVUC = $0.3 \times 850 + \ldots + 0.08 \times 650 = 769.4$

EVPI = EMVUC – EMV = 769.4 – 762.2 = 7.2

**Example:**

Toss fair coin until observe first Tail. If $n$ tosses, win $2^n = x$. How much would we pay to play?

<table>
<thead>
<tr>
<th>Possible values</th>
<th>T</th>
<th>HT</th>
<th>HHT</th>
<th>Etc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
<tr>
<td>$p$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{8}$</td>
<td>...</td>
</tr>
<tr>
<td>$x$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>...</td>
</tr>
</tbody>
</table>

$E[X] = \sum_{n=1}^{\infty} 2^n \times \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty$

Suggests prepared to pay any finite amount to play – not very reasonable!

So i) EMV may not be a reliable guide.

ii) Different people may make different choices in some decision problems.

Example called St. Petersburg paradox (Bernoulli 18th Century), designed to show EMV may not be a good guide to action.

Further example: choose between (1) $3m$ or (2) $8m$ (prob $\frac{1}{2}$), $0$ (prob $\frac{1}{2}$). Many people prefer (1) although (2) has higher EMV.