Section 4: Acceptance-Rejection.

Ensure all previous worksheets are completed before advancing.

1 Sampling from a Normal

In class I explained how we could simulate values from a Normal distribution using the acceptance-rejection technique if we can first simulate values from a uniform and an exponential distribution (go back to your notes to remind yourself if you need to). To simulate 1000 values from a Normal distribution with mean 0 and variance 1 we can do the following (best to use an R-script for this):

```r
x=rep(0,1000) # empty vector to be filled with accepted simulations
k=0 # will keep track of how many simulations we accept
count=0 # will keep track of how many simulations generated
while(k<1000){ # keep proposing until we have accepted 1000 simulations
  count=count+1 # increment proposal counter
  y=rexp(1,1) # simulate a proposal from an exponential distribution with lambda=1
  ratio=exp(-((y-1)^2)/2) # acceptance ratio
  test=runif(1,0,1) # generate acceptance probability
  if(test<ratio){ # acceptance criteria
    k=k+1 # increment acceptance count
  }
  if(runif(1,0,1)<0.5){ # determine if accepted value is positive or negative
    x[k]=y
  } else {
    x[k]=-y
  }
}
```
Task: Using the techniques from the last laboratory worksheet, verify by eye that the simulated values are from the appropriate distribution, \textit{i.e.}, produce a histogram of the simulated numbers and overlay the density function for a standard normal (hint: in \textit{R} this can be given using the \texttt{dnorm(x,0,1)} command within the \texttt{curve} function).

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Task: Using the simulated values in \textit{x}, generate 1000 samples for a Normal distribution with mean 3 and variance 2, and again check by eye that these simulated values appear to follow the correct distribution.

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Task: Determine the efficiency of the simulation, \textit{i.e.}, what proportion of generated proposals are being discarded (recall that by theory, in the limit, this should be $1/c \approx 0.758$)?

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Task: Repeat the simulation using a proposal that is distributed according to an Exponential distribution with parameter $\lambda = 2$ (hint: you will need to change the 5-th and 6-th, lines of the previous code, \textit{i.e.}, the proposal and acceptance ratio lines - and you need to recalculate what the acceptance ratio is - refer to your notes if you are confused):

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Task: Check that the simulated values indeed appear distributed according to a standard normal distribution and verify that the acceptance rate is close to the theoretical limit of $1/c = \sqrt{2\pi}/e^2 \approx 0.339$.

Task: The Beta distribution with parameters $\alpha = 2$ and $\beta = 2$ has p.d.f. $6x(1-x)$ for $x \in [0, 1]$. Use the acceptance-rejection technique to simulate 1000 values from this distribution using a proposal from a Uniform distribution between 0 and 1 (hint: you need to do the math for the acceptance ratio - though its fairly straightforward).
Task: Check that the simulated values indeed appear distributed according to a Beta(2,2) distribution (hint: use `dbeta(x, 2, 2)` in the `curve` function) and verify that the acceptance rate is close to the theoretical limit of $1/c$.

Trickier: A biased die gives a 1 half the time, and the other numbers equally likely otherwise, i.e., $P(X = 1) = 1/2$, while $P(X = 2) = \cdots = P(X = 6) = 1/10$. Simulate 1000 die-rolls using the roll of a fair die as the proposal mechanism, and check the results are as expected (use `table(x)/1000`) and that the efficiency is close to the theoretical level of $1/c = 1/3 \approx 0.333$. (Hint: you will find the `sample` function of use here for your proposal mechanism).