Decision Making with Uncertain Preferences

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Outline:

1. Review of Utility Theory
   - Decision Problems
   - Historical Treatment

2. Alternatives to EU

3. Uncertain Preferences

4. Example
Decision Problems

- A Decision Maker (DM) must select a decision from within a set of available options $\mathcal{D}$.
- Initially there is uncertainty over what will be the resulting outcome of each available decision.
- Let $\mathcal{R}$ be the set of all possible outcomes.
- Selecting a decision is to be seen as equivalent to selecting a distribution over $\mathcal{R}$.

How should the DM select her choice?
Decision Problems

- Many application problems in economics, clinical trials, and reliability *etc.*
- However, also relevant within statistical theory, for example:
  - Estimation.
  - Hypothesis testing.
  - Design of experiments.
Example

A company currently uses service $X$, but has the option of instead using a better service $Y$ for an additional cost of $c_1$. Service $Y$ can either lead to an increase of $s_1$ or $s_2$ in performance, with prior beliefs that either outcome is equally likely.

- Possible outcomes $(c, s)$ display changes to status quo in cost $c$ and performance $s$ respectively.
- Possible decisions are to use service $X$ or to use service $Y$.
- Service $X$ leads to outcome $(0, 0)$ for sure.
- Service $Y$ leads to outcome $(-c_1, s_1)$ with probability 0.5 and outcome $(-c_1, s_2)$ otherwise.
Consider the game where a fair coin is successively flipped.

- For a throw of a head €2 is won and placed in a pot.
- For each further throw of a head the money in the pot is doubled.
- The pot is given to the player as soon as a tail is thrown.
- What is the most you would pay to play this game?
St Petersburg Paradox

- There is \( (1/2)^{n+1} \) chance of winning \( \mathcal{E}2^n \) for \( n = 1, 2, \ldots \).
- The probability of receiving nothing is 0.5.
- The expected return from choosing to play is:

\[
\sum_{n=1}^{\infty} (1/2)^{n+1} \times 2^n = \frac{1}{2} \sum_{n=1}^{\infty} 1 = \infty
\]

Is maximising expected return, à la Pascal’s wager, always a good way to make decisions?
Bernoulli and Utility Hypothesis

- Consider measuring a person’s happiness just like we may do their temperature.
- Rather than in degrees Celsius we do this in *utils*.
- The more utils a person has the happier they are.
- Bernoulli (1738) suggested that giving someone twice as much money will not necessarily give them twice as many utils.
- A utility function $u : \mathcal{R} \rightarrow \mathbb{R}$ need not be linear.

Suggest choose decision maximising expected utility.
Notation

- $d_1 \succeq d_2$ denotes that $d_1$ is at least as preferable as $d_2$.
- $d_1 \succ d_2$ denotes that $d_1$ is strictly preferred to $d_2$.
- $\alpha d_1 + (1 - \alpha)d_2$ denotes the hierarchy in which $d_1$ results with probability $\alpha$, $d_2$ otherwise.
Axioms

- A1 Completeness: $\succeq$ is a complete relation and the set of feasible decisions $\mathcal{D}$ is a closed convex combination of lotteries.

- A2 Transitivity: $\succeq$ is a transitive relation.

A1 requires that a preference relation exists over any two decision choices. A2 prevents ‘money-pump’ situations.
Axioms

- A3 Archimedian: If $d_1, d_2, d_3 \in \mathcal{D}$ are such that $d_1 \succ d_2 \succ d_3$, then there is an $\alpha, \beta \in (0, 1)$ such that:

$$\alpha d_1 + (1 - \alpha) d_3 \succ d_2 \succ \beta d_1 + (1 - \beta) d_3$$

- A4 Independence: For all $d_1, d_2, d_3 \in \mathcal{D}$ and any $\alpha \in [0, 1]$:

$$d_1 \succeq d_2 \iff \alpha d_1 + (1 - \alpha) d_3 \succeq \alpha d_2 + (1 - \alpha) d_3$$

A3 ensures continuity, whilst A4 ensures that the chance of an independent alternative does not affect preferences.
VNM and Utility Theorem

Von Neumann and Morgenstern (1947) showed that axioms A1-A4 are all that is required to prove there exists a unique utility function (up to a positive linear transformation) with the properties that:

1. For all $d_1, d_2 \in D$, $u(d_1) \geq u(d_2) \Leftrightarrow d_1 \succeq d_2$.
2. For all $d_1, d_2 \in D$ and any $\alpha \in (0, 1)$,
   \[ u(\alpha d_1 + (1 - \alpha)d_2) = \alpha u(d_1) + (1 - \alpha)u(d_2). \]

This proves we should choose decision with maximum expected utility return.
Example

In our example all that is needed to determine the solution is the utility function for changes in cost and performance.

- Assume $u[(c, s)] = c + s$.
- Then $u(X) = 0$ and $u(Y) = 0.5(-c_1 + s_1) + 0.5(-c_1 + s_2)$
- Select $Y$ if $0.5(s_1 + s_2) > c_1$ otherwise select $X$. 
Further Extensions for Subjective Probability

- The work of von Neuman and Morgenstern is only applicable if each decision leads to an objective probability distribution over outcomes.
- Savage (1954) proves the expected utility hypothesis for a fully subjective setting.
- Anscombe and Aumann (1963) extend the work of von Neumann and Morgenstern to the subjective setting by adding two additional axioms (Monotonicity and Reversal).
- Further works extend results to continuous distributions, e.g., Herstein and Milnor (1953).
A Which do you prefer out of:

1. €1,000,000 for sure.
2. €1,000,000 with 89% chance, €5,000,000 with 10% chance, €0 with 1% chance.

B Which do you prefer out of:

1. €1,000,000 with 11% chance, €0 with 89% chance.
2. €5,000,000 with 10% chance, €0 with 90% chance.
Allais Paradox

- A preference for A-1 and B-2 has been shown to be prevalent.
- This combination of preferences places two constraints on the utility function that cannot be simultaneously satisfied.
- The problem occurs because the Independence axiom of utility theory is violated.
- The only difference between A-1 and B-1, and between A-2 and B-2, is a common increased chance of receiving €0.
- This is known as the Allais Paradox, named after the French Nobel Prize winner.
- Allais argued against the use of the Independence axiom as a criteria of rationality in decision making.
Alternatives to EU

- There are many suggested alternatives to Expected Utility theory, each of which alters one or more of the axioms of Von Neumann and Morgenstern.

- Allais argued that the entire distribution of ‘utility’ values should be taken into account, not just the expected value.

- There are also many theories which only seek to be descriptive, rather than normative, e.g., Kahneman and Tversky’s Prospect Theory.
• Removing the Completeness axiom is popular in imprecise probability theories, but does not guarantee identification of a best decision, e.g. Maximal (Seidenfeld et al. 1995) or $E$-Admissibility (Levi 1974).

• Other so-called non-Expected Utility theories often disagree with the Independence axiom, e.g., Maximin (Wald 1950).

• Many other possibilities and this is still a very active area of research.
Uncertain Preferences

- The result of von Neumann and Morgenstern only says that there exists a unique utility function, it does not tell us what it is.
- Yet can we always be certain of what our utilities will be?
- Cyert and DeGroot (1975) argued that at times we may be unsure of our utilities (and hence our preferences).
- What effect does this have?
Adaptive Utility

- Adaptive Utility, as introduced by Cyert & DeGroot (1975), considers the use of a parametric family of possible utility functions.
- Leads to a generalisation of Bayesian Statistical Decision Theory, coinciding in the single decision case.
- The unknown utility parameter $\theta$ could be used to represent an unknown trade-off weight or measure of risk-aversion etc.
- Applications of interest may be in Marketing Theory (a new brand becomes available) or Clinical Trials (new drug available) etc.
Reliability and Testing

The effects of adaptive utility upon testing and reliability were considered by Houlding and Coolen (2007):

- Permits subjective cost of system failure to remain uncertain.
- When should systems be fixed if the trade-off between cost of fix and change in performance is unknown?
- Now observing a system failure can be beneficial as, although this implies they occur more frequently, they can be informative of the the uncertain preference relation.
- Decision to replace an unreliable system may be postponed to learn about preference relations.
Effects of Uncertain Utility

- What is the interpretation of a utility parameter and how is it meaningful to compare different utility functions?
- Is there a mathematical argument justifying the maximization of adaptive utility?
- What are the implications for sequential decision selection strategy?
- What are the effects on diagnostics such as value of information and risk aversion?
Interpretation of Utility Parameter

- Let a DM’s *state of mind* $\theta$ characterise the true utility function.
- Assume $\theta$ uncertain so as to represent uncertainty of utility function.
- Each possible value for $\theta$ will induce a different utility function and hence represent different preferences over rewards and decisions.
- Is equivalent to setting a probability distribution over the complete preference relation $\preceq$ over the set of decisions $\mathcal{D}$.
Adaptive Utility Functions

- We now have a set of possible utility functions $u[\cdot|\theta_1], u[\cdot|\theta_2], \ldots$

- Define an adaptive utility function $u_a(\cdot)$ to be expected value of $u[\cdot|\theta]$ with respect to beliefs about $\theta$.

- Formally, $u_a(\cdot) = \mathbb{E}_\theta [u[\cdot|\theta]]$

- Now the utility of a decision will ‘adapt’ according to changes in the belief of $\theta$.

- Repeated application of von Neumann and Morgenstern leads to existence and uniqueness of function.

- Furthermore, a DM should seek to choose decision with maximum expected adaptive utility.
Commensurability

- However, this requires it be meaningful to compare utility values conditioned on differing values of $\theta$.
- A classical utility function is only unique up to a positive linear transformation.
- We need to scale classical utilities so they are commensurable, i.e., so that it is meaningful to make comparisons between terms such as $u(r_1|\theta_1)$ and $u(r_2|\theta_2)$.
- Boutilier (2003) showed this was possible under assumption of extremum equivalence.
- Demonstrated commensurability can even be achieved without extremum equivalence.
Surprise and Selection Strategy

The effects of adaptive utility and accepting preferences are uncertain, but that they may be learned of, are:

- No difference in a one-off decision.
- Can model a situation where a DM is pleasantly surprised by outcome or upset by outcome.
- Depending on length of decision sequence and prior beliefs, DM should select decisions that are likely to lead to unfamiliar outcomes.
- This should be done even if DM expects that outcome will be less favourable than another.
Trial Aversion and Value of Information

The interpretations of utility diagnostics will now be different:

- How much should DM pay for information about likely outcome of a decision if uncertain of preferences?
- No general connection between value of information and level of utility uncertainty, but depends on individual case.
- Trial Aversion is analogous to Risk Aversion (preference or avoidance of actuarially fair bets).
- Consider $u(r|\theta)$ as a function of $\theta$ only.
- Would call DM trial averse in region that $u(r|\theta)$ is concave.
- Purely a diagnostic as never get to choose distribution over $\theta$. 
Return to Example and assume two possible utility functions:

- \( u[(c, s) | \theta] = c + \theta s \)
- \( \theta = \theta_1 = 1 \) with probability 0.5 otherwise \( \theta = \theta_2 = 2 \).
- Assume wish to maximise sum of utility over two periods.
- Assume \( c_1 = 2.3, s_1 = 1 \) and \( s_2 = 2 \) so that in a one off decision DM should stick with service \( X \).
Example

- Information $z$ about $\theta$ comes from observing either positive surprise or negative surprise from utility return after first period.
- Let $z = 1$ for positive surprise and $z = 0$ for negative surprise.
- If service $X$ is chosen utility known for sure and no information gained.
- If service $Y$ is chosen we learn about $\theta$.
- Assume probability that $Y$ truly leads to performance increase $s$ if it was previously seen to do so is 0.9.
- If service $Y$ chosen we thus also learn of its true performance increase.
Influence Diagram
Through use of Bayes’ Theorem we find:

- $P(\theta = 1|z = 1) = 0$
- $P(\theta = 1|z = 0) = 2/3$
- $P(z = 1) = 1/4$
- $P(z = 0) = 3/4$
Example

Through dynamic programming we find:

- Should choose service $Y$ in first period even though under initial beliefs service $X$ seems better.

- Should choose service $Y$ again in second period only if it was seen to lead to the larger increase in performance, otherwise choose service $X$.

- This trivial example is equivalent to solving a decision tree with 25 end nodes.