Examining the Effect of Uncertain Preferences upon Value of Sample Information

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Outline:

Review & Motivation
- Expected Utility Theory
- Value of Information
- Adaptive Utility

Single Decision Problem
- Effect of Uncertain Preferences
- Example
- Information over Preferences

Sequential Decision Problem
- Understanding the Value of a Decision
- Example
Expected Utility Theory

- Let Ω be the set of possible states of nature ω.
- Let \( D \) be a finite set of decisions mapping Ω to a reward set \( \mathcal{R} \).
- Assume true state of nature ω uncertain and denote beliefs by \( P_\omega \).
- Let \( u \) be a cardinal utility function mapping the set \( \mathcal{R} \) to \( \mathbb{R} \).
- Under axioms of expected utility theory, a Decision Maker’s (DM’s) objective is to maximise expected utility return.
- The DM should thus select decision \( d \) maximising \( E_\omega[u(d, \omega)] \).
Let $X$ be unknown information about $\omega$.

Without $X$ DM picks $d$ to maximise $E_\omega[u(d, \omega)]$.

With $X$ DM updates beliefs to $P_{\omega|X}$ and picks $d$ to maximise $E_{\omega|X}[u(d, \omega)]$.

Not knowing $X$ means $E_{\omega|X}[u(d, \omega)]$ is a random variable.

Let $P_X$ represent beliefs over what $X$ will say.

Before $X$ is known, DM expects that knowing $X$ will permit a max expected utility return of $E_X[\max_d E_{\omega|X}[u(d, \omega)]]$. 
Continued...

- Denote the value of $X$ to the DM by $I_\omega(X)$.
- $I_\omega(X)$ represents the fair price (in utility currency) to the DM for knowledge of $X$.
- $I_\omega(X)$ is the max expected return knowing $X$ minus the max expected return not knowing $X$.
- $I_\omega(X)$ is an unknown random variable, its expectation, $E_X[I_\omega(X)]$, is the DM’s expected value of $X$.
- $E_X[I_\omega(X)] = E_X[\max_d E_{\omega|X}[u(d, \omega)]] - \max_d E_\omega[u(d, \omega)]$
Example

- A DM has decision $d$ to either Buy ($B$) or Not Buy ($NB$) stock.
- The state of nature $\omega$ is whether the stock will Rise ($R$) or Fall ($F$), and prior beliefs are that $P_\omega(R) = 0.5$.
- Utility function is such that $u(B, R) = 1$, $u(B, F) = -1$ and $u(NB, R) = u(NB, F) = 0$.
- Max expected utility is thus 0.
Example Continued...

- An expert offers to sell information $X$ on how stock will perform.
- $X$ can be either Good ($G$) or Poorly ($P$).
- You believe the expert is right with chance 0.9, e.g.,
  \[ P_{\omega|X}(R|G) = 0.9. \]
- If $X = G$, max expected return is 0.8, and if $X = P$, then this is 0.
- Calculate predictive to find $P_X(G) = 0.5$.
- Thus $E_X[l_\omega(X)] = (0.5 \times 0.8 + 0.5 \times 0) - 0 = 0.4$. 
Assume DM uncertain of preference ranking over elements in \( \mathcal{R} \).

Let \( \theta \) be the DM’s state of mind representing actual preferences.

A classical utility function is hence conditional on a value of \( \theta \), denoted \( u(r|\theta) \).

An adaptive utility function is a function of both reward and \( \theta \), denoted \( u^a(r, \theta) \).

Once classical utility functions have been scaled to be commensurable, then \( u^a(r, \theta) = u(r|\theta) \).

The DM’s objective is to maximise her adaptive utility function.
Questions

If we start to permit uncertain preferences, then we may be interested in:

- How to generalise EV $X$ for uncertain utility?
- How does EV of $X$ informative of $\omega$ depend on beliefs over $\theta$?
- Can one consider EV of $X$ informative of $\theta$?
- What happens when $X$ only arises following decision selection in sequential problems?
Effect of Uncertain Preferences

Assume beliefs over $\theta$ and $\omega$ independent. If $\theta$ uncertain, and if the objective is to max expected adaptive utility, then the value of $X$ informative of $\omega$ depends on:

- The set $\mathcal{D}$ of possible decisions.
- The adaptive utility function $u^a$.
- Beliefs over $\omega$ as represented by $P_\omega$.
- The likelihood $P_{X|\omega}$.
- But also now on beliefs over $\theta$ as represented by $P_\theta$. 

Brett Houlding, Frank Coolen
Examining the Effect of Uncertain Preferences upon Value of Sample Information
Now the DM is assumed to wish to max her expected adaptive utility function, $E_\omega E_\theta [u(d, \omega, \theta)]$.

If $X$ informative of $\omega$ is known then $\max E_\omega |_XE_\theta [u(d, \omega, \theta)]$.

As before, the expected value of $X$ is the expected max expected utility obtainable with $X$, minus the max expected utility without $X$.

To demonstrate use of adaptive utility, replace $I_y(X)$ notation by $I^a_y(X)$, where subscript $y$ is parameter that $X$ is informative of.

$E_X[I^a_\omega(X)] = E_X[\max_d E_\omega |_XE_\theta [u(d, \omega, \theta)]] - \max_d E_\omega E_\theta [u(d, \omega, \theta)]$
Properties

Similar to case of classical value of information, $E_X [I^a_\omega(X)]$ satisfies:

- **Non-Negativity:** $E_X [I^a_\omega(X)] \geq 0$
- **Additivity:** $E_{X_1,X_2} [I^a_\omega(X_1, X_2)] = E_{X_1,X_2} [I^a_\omega(X_2|X_1)] + E_{X_1} [I^a_\omega(X_1)]$
Relationship between Uncertainty and Value

Less uncertainty over $\theta$ means the DM is more sure over preferences, so one may guess that more uncertainty decreases $E_X[I_{\omega}(X)]$. However, there appears to be no simple relationship between uncertainty over $\theta$ and $E_X[I_{\omega}(X)]$. In particular:

- $E_X[I_{\omega}(X)]$ is not necessarily increased as less values of $\theta$ are possible.
- $E_X[I_{\omega}(X)]$ is not necessarily increased as less probability is concentrated on a particular value of $\theta$.
- $E_X[I_{\omega}(X)]$ is not necessarily increased as $V_\theta[\theta]$ decreases.
- $E_X[I_{\omega}(X)]$ is not necessarily minimised in the case of equiprobable values of $\theta$. 
Example

The previous statements can be shown to be a result of the following counter example:

- $\theta \in \{\theta_1, \theta_2\}$, $P_\theta(\theta_1) = p$
- $\omega \in \{\omega_1, \omega_2\}$, $P_\omega(\omega_1) = 0.5$
- $X \in \{x_1, x_2\}$, $P_X|\omega(x_j|\omega_1) = \delta_{ij}$
- $d \in \{d_1, d_2\}$, $d_1(\omega_1) = d_2(\omega_2) = r_1$, $d_1(\omega_2) = d_2(\omega_1) = r_2$
- $u^a(r_1, \theta_1) = u^a(r_2, \theta_1) = 1$, $u^a(r_1, \theta_2) = 2$, $u^a(r_2, \theta_2) = 0$
Continued...

We find that:

- \( \max_d E_\omega E_\theta [u^a(d, \omega, \theta)] = 1, \forall p \in [0, 1] \)
- \( E_X[\max_d E_\omega|X E_\theta[u^a(d, \omega, \theta)]] = 2 - p \)
- \( E_X[l^a_\omega(X)] = 1 - p \)
- Setting \( \theta_1 = v_1 \) and \( \theta_2 = v_2 \) gives \( V_\theta[\theta] = p(1 - p)(v_1 - v_2)^2 \).

For a change in \( P_\theta \) to increase \( E_X[l^a_\omega(X)] \) it must increase \( E_X[\max_d E_\omega|X E_\theta[u^a(d, \omega, \theta)]] \) more than \( \max_d E_\omega E_\theta[u^a(d, \omega, \theta)] \).
Dependencies & Information over $\theta$

- No problem in dealing with dependency of beliefs over $\theta$ and $\omega$, and similar results and properties hold.
- Information over $\theta$ could arise from interview with experts.
- It could also arise through trial of rewards, e.g., test drive or sample tasters, or from observation of Advertisements.
- Usually, however, we envisage $X$ informative of $\theta$ will be observed following decision selection in a sequential problem.
Understanding the Value of a Decision

- Now consider period $i$ that $X$ is observed: $I_Y^a(X; i)$.
- $I_Y^a(X; i)$ can not increase if $i$ is increased.
- If $d^i$ selected in period $i$ influences $X$, then must consider $E_X[I_{\omega}^a(X; i)]$ in connection with $d^i$.
- Now decompose total expected value of decision $d^i$ into expected ‘pure’ value (resulting from the reward it entails), and expected ‘information’ value (resulting from value of $X$).
- This is why a DM in a sequential problems may select decisions not optimal under prior beliefs, and that such experimentation should occur at the earliest opportunity.

Brett Houlding, Frank Coolen

Examining the Effect of Uncertain Preferences upon Value of Sample Information
Example

- Must choose an Apple (A) or Banana (B) in each of 3 periods.
- Assume $\omega$ redundant (a decision to choose $A$ will result in $A$).
- $u^a(d^1, d^2, d^3, \theta) = \sum_{j=1}^{3} I\{d^i=A\} + \theta \sum_{j=1}^{3} I\{d^i=B\}$
- Thus choosing $A$ increases utility by 1, $B$ increases this by $\theta$.
- Assume $\theta \in \{0.5, 1.5\}$ and $P_\theta(0.5) = 0.6$.
- Let $X = 1$ if DM chooses $A$, whilst $X = \theta$ if DM chooses $B$.
- Consider value of $X$ observed after $d^1$. 
Continued...

- \( \mathbb{E}_X[I^a_{\theta}(X;1)] = 0 \) for \( d^1 = A \). No information can be gained!
- \( \mathbb{E}_X[I^a_{\theta}(X;1)] = 0.3 \) for \( d^1 = B \). We will learn \( \theta \) for sure!
- Thus, even though selecting Apple gives sure reward of value 1, whilst prior expectation of value resulting from a Banana is \( E_\theta[\theta] = 0.9 \), \( d^1 = B \) has extra value of 0.3 because of its expected information.
- Hence why a DM does best in expectation by first selecting \( d^1 = B \).