On Separation, Session Types and Algebra

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Formalisms

Modularity

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Session Types

- Process Calculi
- Message Passing

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Concurrent Separation Logic

- Imperative Programs
- Shared Resource

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Algebra

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Baby Session Types

Programs

$$P \quad ::= \quad k?j.P \mid k!j.P \mid P \parallel P \mid \texttt{inact}$$

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$$\alpha, \beta \ ::= ![\alpha];\beta \mid ?[\alpha];\beta \mid \texttt{end}$$

Co-Types

$$\overline{![\alpha];\beta} = ?[\alpha];\overline{\beta} \quad \overline{?[\alpha];\beta} = ![\alpha];\overline{\beta} \quad \overline{\texttt{end}} = \texttt{end}$$

Baby Session Types

Typing Context

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Typing

$$P \triangleright \Delta$$

Proof Rules for BST

$$[\text{Consequence}]\ \frac{\Delta_1 \vdash \Delta_2 \quad P \triangleright \Delta_2}{P \triangleright \Delta_1}$$

$$[\text{Inact}]\ \frac{}{\texttt{inact} \triangleright \emptyset} \qquad [\text{Par}]\ \frac{P_1 \triangleright \Delta_1 \quad P_2 \triangleright \Delta_2}{P_1 \parallel P_2 \triangleright \Delta_1 \circ \Delta_2}$$

$$[\text{Receive}]\ \frac{P \triangleright \Delta \circ k : \beta \circ j : \alpha}{k?j.P \triangleright \Delta \circ k : ?[\alpha]; \beta} \qquad [\text{Send}]\ \frac{P \triangleright \Delta \circ k : \beta}{k!j.P \triangleright \Delta \circ k : ![\alpha]; \beta \circ j : \alpha}$$

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Examples

Example (1 - Racey programs do not type check)

$$(k!x.\texttt{inact}) \parallel (k!y.\texttt{inact}) \triangleright \Delta$$

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Example (2 - Ownership Transfer)

$$\texttt{Let } H = ![\alpha];\texttt{end}$$

Process 1	Process 2	Process 3
$\{h : H,\ h' : \overline{H},\ k :![H];\texttt{end},\ j :![\overline{H}];\texttt{end}\}$	$\{k :?[H];\texttt{end},\ w : \texttt{end}\}$	$j :?[\overline{H}];\texttt{end}\}$
$k!h$	$k?(x)$	$j?(y)$
$\{h' : \overline{H},\ k : \texttt{end},\ j :![\overline{H}];\texttt{end}\}$	$\{x : H,\ k : \texttt{end},\ w : \texttt{end}\}$	$\{y : \overline{H},\ j : \texttt{end}\}$
$j!h'$	$x!w$	$y?(z)$
$\{k : \texttt{end},\ j : \texttt{end}\}$	$\{k : \texttt{end},\ x : \texttt{end}\}$	$\{j : \texttt{end},\ y : \texttt{end},\ z : \texttt{end}\}$

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Process 1	*Process* 2	*Process* 3

Outline

Basic Concurrent Separation Logic

A preordered commutative monoid of propositions

$$(\mathit{Props}, \vdash, *, \mathit{emp})$$

A set of commands (*Com*)

Equipped with total binary operations $c \parallel c'$ and $c; c'$ with $\mathit{skip} \in \mathit{Com}$

Proof Rules for BCSL

$$[\text{Skip}]\ \frac{}{\{X\}\ \texttt{skip}\ \{X\}} \qquad [\text{Frame}]\ \frac{\{X\}\ c\ \{Y\}}{\{X * F\}\ c\ \{Y * F\}}$$

$$[\text{Seq}]\ \frac{\{X\}\ c_1\ \{Y\} \quad \{Y\}\ c_2\ \{Z\}}{\{X\}\ c_1 ; c_2\ \{Z\}} \qquad [\text{Par}]\ \frac{\{X_1\}\ c_1\ \{Y_1\} \quad \{X_2\}\ c_2\ \{Y_2\}}{\{X_1 * X_2\}\ c_1 \parallel c_2\ \{Y_1 * Y_2\}}$$

$$[\text{Consequence}]\ \frac{X' \vdash X \quad \{X\}\ c\ \{Y\} \quad Y \vdash Y'}{\{X'\}\ c\ \{Y'\}}$$

Parallel Rules

$$[\text{BST}]\quad \frac{P_1 \triangleright \Delta_1 \quad P_2 \triangleright \Delta_2}{P_1 \parallel P_2 \triangleright \Delta_1 \circ \Delta_2}$$

$$[\text{BCSL}]\quad \frac{\{X_1\}\, c_1\, \{Y_1\} \quad \{X_2\}\, c_2\, \{Y_2\}}{\{X_1 * X_2\}\, c_1 \parallel c_2\, \{Y_1 * Y_2\}}$$

Heap Model Instantiation

Structure of propositions

$$(\mathit{Props}, \vdash, *, \mathit{emp}) = (P(\mathit{Heaps}), \subseteq, *, \{u\})$$

- Heaps: $\mathbb{N} \rightharpoonup_f \mathbb{N}$
- $P(\mathit{Heaps})$: Powerset
- u: Empty partial function.
- $X * Y = \{h_X \bullet h_Y \mid h_X \in X \wedge h_Y \in Y \wedge h_X \bullet h_Y \downarrow\}$ where $h \bullet h'$ denotes the union of disjoint heap.

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$$\frac{}{\{n \mapsto -\}[n] := m\{n \mapsto m\}}$$

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$$[10] := 23 \parallel [10] := 44$$

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$$10 \mapsto - \ * \ 10 \mapsto -$$ is *false*

Example (2 - Ownership Transfer via Shared Buffer)

$$\begin{array}{c}
\{emp\} \\
\{emp * emp\} \\
\begin{array}{c|c}
\{emp\} & \{emp\} \\
x := \texttt{cons(a,b)}; & \mathit{getWhenFull}(y); \\
\{x \mapsto -, -\} & \{y \mapsto -, -\} \\
\mathit{putWhenEmpty}(x); & \mathit{use}(y); \\
\{emp\} & \{y \mapsto -, -\} \\
 & \mathit{dispose}(y) \\
\{emp\} & \{emp\}
\end{array} \\
\{emp * emp\} \\
\{emp\}
\end{array}$$

Outline

Algebra for Concurrency (Hoare et al 2009)

- Two ordered monoids $(S, \sqsubseteq, *, u)$ and $(S, \sqsubseteq, ;, \mathit{skip})$ representing parallel and sequential composition, where $*, ;$ are montone and $*$ is commutative.
- Parallel and Sequencing are related by the Exchange Law

$$(p * r); (q * s) \sqsubseteq (p; q) * (r; s) \quad p, q, r, s \in S$$

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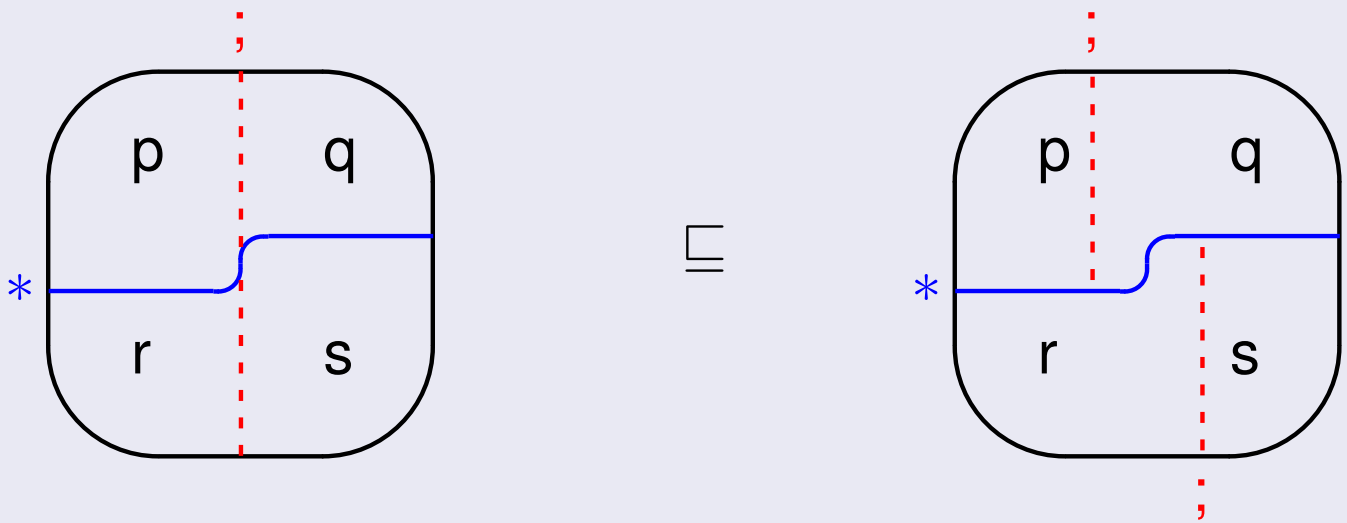

Exchange Law

Validates Plotkin Triple (to come)

- Concurrency Rule
- Frame Rule (when $P * \mathit{skip} = P$)

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$\{P\}\, C\, \{Q\} \Leftrightarrow P \sqsupseteq C; Q$

Proof:

$$
\begin{array}{lll}
 & P \sqsupseteq C; Q \ \wedge \ P' \sqsupseteq C'; Q' & \\
\Rightarrow & P * P' \sqsupseteq (C; Q) * (C'; Q') & \texttt{monotonicity of} * \\
\Rightarrow & P * P' \sqsupseteq (C * C'); (Q * Q') & \texttt{exchange Law}
\end{array}
$$

■

Outline

Structure

$(Props, \vdash, *, emp)$

- *Props* to be the set of session typing contexts Δ
- $\Delta * \Delta'$ to be $\Delta \circ \Delta'$
- *emp* is the empty context $\emptyset$
- $X \vdash Y$ where $X \vdash Y$ *iff* X is inconsistent or $\exists \Phi.\ X = Y \circ \Phi$

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Commands

$$C ::= k?j.C \mid k!j \mid C \parallel C \mid C; C \mid \texttt{skip}$$

Specialised Rules for Session Instantiation

$$[\text{Send}] \quad \frac{}{\{k : ![\alpha]; \beta * j : \alpha\}\ k!j\ \{k : \beta\}}$$

$$[\text{Receive}] \quad \frac{\{A * k : \beta * j : \alpha\}\ P\ \{B\}}{\{A * k : ?[\alpha]; \beta\}\ k?j.P\ \{B\}}$$

Outline

Translation

BST to BSCL

$$
\begin{aligned}
\langle\!\langle \texttt{inact} \rangle\!\rangle &= \texttt{skip} \\
\langle\!\langle P \parallel Q \rangle\!\rangle &= \langle\!\langle P \rangle\!\rangle \parallel \langle\!\langle Q \rangle\!\rangle \\
\langle\!\langle k?j.P \rangle\!\rangle &= k?j.\langle\!\langle P \rangle\!\rangle \\
\langle\!\langle k!j.\, P \rangle\!\rangle &= (k!j)\,;\, \langle\!\langle P \rangle\!\rangle
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\end{aligned}
$$

Theorem 1 - Soundness & Completeness

$P \triangleright \Delta$ is provable in **BST** if and only if $\{\Delta\}\ \langle\!\langle P \rangle\!\rangle\ \{emp\}$ is provable in **BCSL/ST**

Outline

Structure

Propositions

Suppose we have an ordered total commutative monoid $(\mathit{Props}, \vdash, *, \mathit{emp})$ with a least element $\bot$

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Predicates

- Model built from predicate transformers on non-empty down-wards closed subsets of *Props* (*Preds*).
- $(\mathit{Preds}, \subseteq)$ has a total commutative monoid structure $(\mathit{Preds}, \subseteq, \otimes, I)$

$$\begin{array}{lcl} X \otimes Y & = & \{p \mid p \vdash x * y \wedge x \in X \wedge y \in Y\} \\ I & = & \{p \mid p \vdash \mathit{emp}\} \end{array}$$

Structure

Commands

Montone functions space $\mathit{Preds} \to \mathit{Preds}$

$$
\begin{aligned}
(F \parallel G)X &= \bigcup\{FX_1 \otimes GX_2 \mid X_1 \otimes X_2 \subseteq X\} \\
\texttt{nothing}X &= \texttt{if } X \supseteq I \texttt{ then } I \texttt{ else false} \\
(F; G)X &= F(G(X)) \\
\texttt{skip}X &= X
\end{aligned}
$$

$X \in \mathit{Preds}$

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$X \in \mathit{Preds}$

Order

$$F \sqsubseteq G \iff \forall X.\, FX \supseteq GX.$$

Algebraic Structure

Monoids

$(\mathit{Preds}, \sqsubseteq, \|, \texttt{nothing})$ and $(\mathit{Preds}, \sqsubseteq, ;, \mathit{skip})$ form monoids where $\|, ;$ are monotone and $\|$ is commutative.

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Exchange Law

The predicates transformers satisfy

$$(F_1 \parallel F_2); (G_1 \parallel G_2) \sqsubseteq (F_1; G_1) \parallel (F_2; G_2)$$

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Futuristic pre/post spec in Algebra

Plotkin Triple

$$\{P\}\, C\, \{Q\} \quad \Longleftrightarrow \quad P \sqsupseteq C; Q$$

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Session Types

$$\{\Delta\}\, C\, \{\Delta'\} \quad \Longleftrightarrow \quad [\![\Delta]\!] \sqsupseteq [\![C]\!]; [\![\Delta']\!]$$

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Predicate Transformer Dijkstra Triple

$$\langle Y \rangle F \langle Z \rangle \quad \Longleftrightarrow \quad Y \subseteq FZ$$

Futuristic pre/post spec in Algebra

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Predicate Transformer Dijkstra Triple

$$\langle Y \rangle F \langle Z \rangle \quad \Longleftrightarrow \quad Y \subseteq FZ$$

Predicate Transformer Plotkin Triple

$$trans[Y] \sqsupseteq F; trans[Z]$$

Dijkstra & Plotkin Triples

Theorem 2: Predicate Transformers and Proof Theory Agree

Assuming that a local and monotone predicate transformer $[\![c_{prim}]\!]$ is given for a collection of primitive commands, then

$$p \in [\![c]\!]X \iff \exists q \in X.\ \{p\}\, c\, \{q\}$$

holds for all c, as long as it holds for primitive commands.

Dijkstra & Plotkin Triples

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Theorem 3: Predicate Transformers and Algebra Agree

For all $Y, Z \in \mathit{Preds}$ and monotone $F : \mathit{Preds} \to \mathit{Preds}$,

$$Y \subseteq FZ \iff \mathit{trans}[Y] \sqsupseteq F; \mathit{trans}[Z]$$

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Questions...

www.eecs.qmul.ac.uk/∼akbar/OnSeparationSessionTypesAlgebra.pdf

Exchange Law Proof

$$(F_1 \parallel F_2);(G_1 \parallel G_2) \sqsubseteq (F_1;G_1) \parallel (F_2;G_2)$$

The validity of the exchange law can be seen from the following calculation.

$$\begin{array}{rl} & ((F_1 \parallel F_2);(G_1 \parallel G_2))X \\ = & \bigcup\{F_1 Y_1 \otimes F_2 Y_2 \mid Y_1 \otimes Y_2 \subseteq (G_1 \parallel G_2)X\} \\ = & \bigcup\{F_1 Y_1 \otimes F_2 Y_2 \mid Y_1 \otimes Y_2 \subseteq \bigcup\{G_1 X_1 \otimes G_2 X_2 \mid X_1 \otimes X_2 \subseteq X\}\} \\ \supseteq & \bigcup\{F_1(G_1 X_1) \otimes F_2(G_2 X_2) \mid X_1 \otimes X_2 \subseteq X\} \\ = & \bigcup\{(F_1;G_1)X_1 \otimes (F_2;G_2)X_2 \mid X_1 \otimes X_2 \subseteq X\} \\ = & ((F_1;G_1) \parallel (F_2;G_2))X \end{array}$$

In the $\supseteq$ step we take $Y_1 = G_1 X_1$, $Y_2 = G_2 X_2$. This step uses that $X_1 \otimes X_2 \subseteq X \Rightarrow G_1 X_1 \otimes G_2 X_2 \subseteq \bigcup\{G_1 X_1 \otimes G_2 X_2 \mid X_1 \otimes X_2 \subseteq X\}$.

Predicate Converter (Trans)

$$\mathit{do\text{-}after}[Y]X = \texttt{if}\ X = \mathit{Props}\ \texttt{then}\ Y\ \texttt{else}\ \mathsf{false}$$